NASA CR-166,110

NASA Contractor Report 166110

ICASE

NASA-CR-166110
19830020668

N-PERSON DIFFERENTIAL GAMES
PART I: DUALITY-FINITE ELEMENT METHODS

Goong Chen

and

Quan Zheng

Contract No. NAS1-15810
April 1983

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

N-PERSON DIFFERENTIAL GAMES
PART I: DUALITY-FINITE ELEMENT METHODS

Goong Chen* and Quan Zheng**
Pennsylvania State University

ABSTRACT

Standard theory of differential games focuses the study on two-person zero-sum games, and treat N-person games separately and differently. In this paper we present a new equivalent formulation of the Nash equilibrium strategy for N-person differential games. Our contributions are the following:

1) Our min-max formulation unifies the study of two-person zero-sum with that of the general N-person non zero-sum games. Indeed, it opens a new avenue of systematic research for differential games.
2) We are successful in applying the finite element method to compute solutions of linear-quadratic N-person games. We have also established numerical error estimates. Our calculations, which are based upon the dual formulation, are very efficient.
3) We are able to establish global existence and uniqueness of solutions of the Riccati equation in our form, which is important in synthesis. This, to our knowledge, has not been done elsewhere by any other researchers.

This paper's particular emphasis is on the duality approach, which is motivated by computational needs and is done by introducing N + 1 Language multipliers: one for each player and one "joint multiplier" for all players. For N-person linear quadratic games, we show that under suitable conditions the primal min-max problem is equivalent to its dual min-max problem, which is actually a saddle point and is then computed by finite elements. Numerical examples are presented in the last section.

---

*Department of Mathematics, Pennsylvania State University, University Park, PA 16082. Supported in part by NSF Grant MCS 81-01892 and NASA Contract No. NAS1-15810, the latter while the first author was in residence at the Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA 23665.
**Department of Mathematics, Pennsylvania State University, University Park, PA 16802. Permanent address: Department of Mathematics, Shanghai University of Science and Technology, Shanghai, China.

§0. Introduction

Consider an N-person differential game with linear dynamics

$$(0.1)\qquad \begin{cases} \dfrac{d}{dt}x(t) = A(t)x(t) + B_1(t)u_1(t) + \dots + B_N(t)u_N(t) + f(t), \quad 0 \le t \le T, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases}$$

where $u_i \in U_i \equiv L^2_{m_i} \equiv L^2(0,T;\mathbb{R}^{m_i})$ is the control variable under the command of the i-th player P_i; A, B_i are proper $n \times n$, $n \times m_i$ matrix valued functions, $f \in L^2_n \equiv L^2(0,T;\mathbb{R}^n)$ is the inhomogeneous term and x is the state variable.

An N-tuple of controls $u = (u_1,\dots,u_N) \in U \equiv \prod_{i=1}^{N} U_i$ is called an open-loop strategy. Associated with each player P_i is a cost functional $J_i(x,u)$ $(1 \le i \le N)$ incurred in a game due to a strategy u and the outcome x of (0.1) that is generated by u. The case when J_i is quadratic of the form (3.1) in §3 will be of particular interest to us.

Each player P_i wishes to minimize his cost J_i. We say that $\hat{u} = (\hat{u}_1,\hat{u}_2,\dots,\hat{u}_N)$ forms an (optimal) <u>equilibrium strategy</u> if

$$(0.2)\qquad J_i(x,\hat{u}_1,\dots,\hat{u}_N) \le J_i(x,\hat{u}_1,\dots,\hat{u}_{i-1},v_i,\hat{u}_{i+1},\dots,\hat{u}_N), \quad 1 \le i \le N,$$

for all $v_i \in U_i$. Such a strategy allows all players to play individual optimal strategies simultaneously. Therefore the questions of its existence, uniqueness, solutions and computations constitute the most important study in the theory of N-person differential games.

Standard theory of differential games (e.g. [6],[8]) focus the study on two-person zero-sum games, and treat N-person games separately and differently. For two person zero-sum games, the concept of an equilibrium strategy coincides with that of a saddle point. One then proceeds to use either the Pontryagin (Friedman, Issacs) minimaximum principle or the Bellman dynamic programming to derive necessary conditions for equilibrium. To compute optimal strategies, one must either solve (usually) a two-point boundary value problem of ODEs or a PDE (the Bellman-Hamilton-Jacobi equation).

For N-person games, the pioneering work was done by Lukes and Russell [9]. Their basic point of view, which was inherited in most of the subsequent papers on this subject, actually was to regard N-person differential games as a more complex N-simultaneous optimization problem. From (0.2), they regard $\hat{u}_i$ as the optimal control for the i-th player when other players are using respective strategies $\hat{u}_1, \ldots, \hat{u}_{i-1}, \hat{u}_{i+1}, \ldots \hat{u}_N$. So they proceeded to use the primal, dual, or feedback synthesis methods to solve

$$\begin{cases} \underset{v_i}{\text{Min}} \; J_i(x, \hat{u}_1, \ldots, \hat{u}_{i-1}, v_i, \hat{u}_{i+1}, \ldots, \hat{u}_N) \\ \text{subject to} \\ \dot{x} = Ax + \sum\limits_{j \neq i} B_j \hat{u}_j + B_i v_i + f \\ x(0) = x_0 \end{cases}$$

for players $i = 1,2,\ldots,N$. So $\hat{u}_i$ can be obtained by differentiating J_i with respect to v_i, while holding other players' individual optimal strategies fixed. This yields N simultaneous equations for $\hat{u}_1, \ldots, \hat{u}_N$. The solvability of these equations gives <u>N necessary conditions</u> in general. Even if these N equations

can be solved simultaneously, it is not certain (except perhaps in the linear-quadratic case, wherein the invertibility of certain operators is a sufficient condition) that the derived controls $\hat{u}_1,\ldots,\hat{u}_N$ indeed form an equilibrium strategy, since $\hat{u}_1,\ldots,\hat{u}_N$ mutually interfere through the system dynamics. As a matter of fact, the game-theoretic nature of the problem seems to be lost in this approach.

In this paper, we present a new approach to N-person games - we show that an N-person game can also be formulated into a min-max point problem (§1). This formulation gives a necessary and sufficient condition for the existence of equilibrium strategies. This min-max problem is primal. Later on, we will see that under certain conditions this min-max problem is actually a saddle point problem. In this sense, we see that our work has unified the theory of two-person zero-sum games with the theory of N-person non zero-sum games.

In §2, we formulate the dual of the primal problem, which becomes a max-min problem. In the dual formulation, system dynamical equations like (0.1) are eliminated, thus the new max-min problems is unconstrained. The dual problem is formulated in terms of N+1 Lagrange multipliers $p_i (0 \le i \le N)$: one multiplier p_i for each player P_i $(1 \le i \le N)$ and one "joint multiplier" p_0 for all players.

Beginning from §3, we specialize to the quadratic cost case. We formally synthesize the closed-loop equilibrium strategy and derive the (new) Riccati equation (3.13) which is different from those in other formulations (see e.g. [6], [9]).

§4 deals with the variational formulation of the dual problem. Here we make several assumptions which ensure the tractability of the dual problem. Then the "primal-dual equivalance theorem" is established. The important existence and uniqueness of equilibrium strategy is proved in Theorem 4.7.

In §5, we establish the global existence and uniqueness of the solution of the Riccati equation.

§6 studies finite element approximations. Our work here is motivated by similar work on the Ritz-Trefftz and the finite element methods for optimal controls (see, e.g. [2],[10]). To our knowledge, this is the first time the finite element method is applied to differential games.

Numerical results are given in the last § 7.

In our sequel, Part II [3], we will again use the basic formulation in §1, but combine it with the <u>penalty</u> and the finite element methods, and compare our numerical results from these different approaches.

§1. Equilibrium Strategy as Min-Max Point

We first formulate a sufficient condition which states that an equilibrium strategy can be found as a min-max point. In a two-person zero-sum game, such a saddle point formulation is given a priori. However, for an N-person game our formulation seems to be completely new; it forms the basis for all of our future discussions.

For each $u \in U$, one can solve x from (0.1) and determine $J_i(x,u)(1 \le i \le N)$. Thus each $J_i(x,u)$ is a functional on $(u_1,\ldots,u_N)$, so we define

$$(1.1) \qquad \ell_i(u_1,\ldots,u_n) \equiv J_i(x,u_1,\ldots,u_N).$$

For $u,v \in U$, $u = (u_1,\ldots,u_N)$, $v = (v_1,\ldots,v_N)$, let

$$(1.2) \qquad F(u,v) \equiv \sum_{i=1}^{N} [\ell_i(u) - \ell_i(v^i)], \quad v^i \equiv (u_1,\ldots,u_{i-1},v_i,u_{i+1},\ldots,u_N).$$

<u>Lemma 1.1</u> If $u^* = (u_1^*,\ldots,u_N^*)$ satisfies

$$(1.3) \qquad \sup_{v \in U} F(u^*,v) \le 0,$$

then u^* is an equilibrium strategy. Conversely, if u^* is an equilibrium strategy, then (1.3) holds.

<u>Proof</u>: Assume that (1.3) holds. Choose $v^i = (u_1^*,\ldots,u_{i-1}^*,v_i,u_{i+1}^*,\ldots,u_N^*)$, where $v_i \in U_i$ is arbitrary. Then

$$(1.4) \qquad F(u^*,v^i) \le \sup_{v \in U} F(u^*,v) \le 0.$$

But

$$F(u^*,v^i) = \ell_i(u_1^*,\ldots,u_N^*) - \ell_i(u_1^*,\ldots,u_{i-1}^*,v_i,u_{i+1}^*,\ldots,u_N^*)$$

which is less than or equal to 0 by (1.4). So (0.2) is satisfied; u^* is an equilibrium strategy.

Conversely, if u^* is an equilibrium strategy, then

$$\ell_i(u_1^*,\ldots,u_N^*) - \ell_i(u_1^*,\ldots,u_{i-1}^*,v_i,u_{i+1}^*,\ldots,u_N^*) \le 0\,,\ \forall v_i \in U_i. \tag{1.5}$$

Summing (1.5) from 1 through N, we get $F(u^*,v) \le 0\,, \forall v \in U$. Hence (1.3) holds. □

<u>Theorem 1.2</u> If

$$\inf_{u\in U}\ \sup_{v\in U} F(u,v) < 0 \tag{1.6}$$

or

$$\min_{u\in U}\ \sup_{v\in U} F(u,v) \le 0 \tag{1.6'}$$

is satisfied, then the differential game has at least one equilibrium strategy.

<u>Proof</u>: Under (1.6), we have at least one $\bar{u} \in U$ such that $\sup F(\bar{u},v) \le 0$ $\forall v \in U$. By Lemma 1.1, $\bar{u}$ is an equilibrium strategy. Same conclusion holds for (1.6'). □

Remark 1.3 In the above proof, we see that if we choose $v = \bar{u}$, then

$$0 = F(\bar{u},\bar{v}) \le \sup_{v} F(\bar{u},v) \le 0,$$

therefore $\sup_{v} F(\bar{u},v) = 0$. We see that it is impossible to have $\sup_{v} F(u,v) < 0$. Thus (1.6) is ruled out. An equilibrium strategy exists if and only if

$$\text{(1.6")} \qquad \min_{u \in U} \sup_{v \in U} F(u,v) = 0.$$

A simple corollary is that if $(\bar{u},\bar{v})$ solves

$$\text{(1.7)} \qquad F(\bar{u},\bar{v}) = \min_{u \in U} \max_{v \in U} F(u,v) = 0,$$

then $\bar{u}$ is an equilibrium strategy. □

Remark 1.4 In the discussion above, nowhere have we used the linear dynamics of (0.1). Therefore Theroem 1.2 and Remark 1.3 are valid under the general setting of [6].

Therefore, the question of finding an equilibrium strategy is reduced to solving the min-max problem (1.7) or (1.6").

From now on, we signify the Sobolev space

$$H_n^k \equiv H_n^k(0,T) \equiv \{y : [0,T] \to \mathbb{R}^n \mid \|y\|_{H_n^k} \equiv \sum_{j=0}^{k} \|(\tfrac{d}{dt})^j y\|_{L_n^2} < \infty\}.$$

We define

$$J(x,u;X,v) \equiv J(x,u_1,\ldots,u_N;x^1,\ldots,x^N,v_1,\ldots,v_N)$$

$$\equiv \sum_{i=1}^{N} [J_i(x\ ,u_1,\ldots,u_N) - J_i(x^i,u_1,\ldots,u_{i-1},v_i,u_{i+1},\ldots,u_N)],$$

where $X = (x^1,\ldots,x^N) \in [H_n^1]^N$ and each x^i is the solution of

$$(1.8)\quad \begin{cases} \dot{x}^i = Ax^i + B_1u_1 + \ldots + B_{i-1}u_{i-1} + B_iv_i + B_{i+1}u_{i+1} + \ldots + B_Nu_N + f \\ x^i(0) = x_0. \end{cases}$$

If the given differential game has at least one equilibrium strategy, then we can consider solving

$$(1.9)\qquad \min_{\substack{x,u \\ (DE)=0}} \ \max_{\substack{X,v \\ [DE]=0}} \ J(x,u;X,v),$$

where

$$(1.10)\qquad (DE) = \dot{x} - Ax - \sum_{j=1}^{N} B_ju_j - f, \quad x \in H_n^1, \text{ subject to } x(0) = x_0,$$

$$(1.11)\qquad [DE] = \sum_{i=1}^{N} |(DE)_i|^2 , \quad X(0) = (x^1(0),\ldots,x^N(0)) = (x_0,\ldots,x_0) \equiv X_0,$$

and

$$(1.12)\quad (DE)_i \equiv \dot{x}^i - Ax^i - \sum_{\substack{j=1 \\ j\neq i}}^{N} B_ju_j - B_iv_i - f,\ x^i \in H_n^1, \text{ subject to } x^i(0) = x_0.$$

Suppose that the cost functional is given as

$$J_i(x,u) = \int_0^T h_i(t,x(t),u_1(t),\ldots,u_N(t))dt + g_i(x(T)).$$

In our framework, we can define the Hamiltonian as

$$\begin{aligned}(1.13)\qquad H(t,x,u,X,v,q_0,q) \equiv \sum_{i=1}^{N} [&h_i(t,x(t),u_1(t),\ldots,u_N(t)) - h_i(t,x^i(t),\\ &u_1(t),\ldots,u_{i-1}(t),v_i(t),u_{i+1}(t),\ldots,u_N(t))]\\ &+ \langle q_0(t),A(t)x(t) + \sum_{i=1}^{N} B_i(t)u_i(t) + f(t)\rangle\\ &+ \sum_{i=1}^{N} \langle q_i(t),A(t)x^i(t) + \sum_{\substack{j=1\\ j\neq i}}^{N} B_j(t)u_j(t) + B_i(t)v_i(t) + f(t)\rangle,\end{aligned}$$

where $q = (q_1,q_2,\ldots,q_N)$. The Pontryagin minimaximum principle can be stated as follows: Assume that $(\hat{u},\hat{v})$ is a min-max point for $\min_u \max_v F(u,v)$ subject to (DE) = 0, [DE] = 0; let $\hat{x},\hat{X},\hat{q}_0,\hat{q}$ satisfy the canonical equations

$$(1.14)\qquad \frac{d\hat{x}(t)}{dt} = \frac{\partial}{\partial q_0} H(t,\hat{x},\hat{u},\hat{X},\hat{v},q_0,\hat{q})\Big|_{q_0=\hat{q}_0} \quad ; \quad \hat{x}(0) = x_0,$$

$$(1.15)\qquad \frac{d\hat{x}^i(t)}{dt} = \frac{\partial}{\partial q_i} H(t,\hat{x},\hat{u},\hat{X},\hat{q}_0,q)\Big|_{q=\hat{q}} \quad ; \quad \hat{x}^i(0) = x_0, \quad 1 \le i \le N,$$

$$(1.16)\qquad \frac{d\hat{q}_0(t)}{dt} = -\frac{\partial}{\partial x} H(t,x,\hat{u},\hat{X},\hat{v},\hat{q}_0,\hat{q})\Big|_{x=\hat{x}} \quad ; \quad \hat{q}_0(T) = \sum_{i=1}^{N} \frac{\partial}{\partial x} g_i(x)\Big|_{x=\hat{x}(T)},$$

$$(1.17)\qquad \frac{d\hat{q}_i(t)}{dt} = -\frac{\partial}{\partial x^i} H(t,\hat{x},\hat{u},X,\hat{v},\hat{q}_0,\hat{q})\Big|_{X=\hat{X}}\ ;\ \hat{q}_i(T) = -\frac{\partial}{\partial x^i} g_i(x^i)\Big|_{x^i=\hat{x}^i(T)}.$$

Then, we have, necessarily, the Hamiltonian at a min-max point for all time $t \in (0,T)$:

$$(1.18)\qquad H(t,\hat{x},\hat{u},\hat{X},\hat{v},\hat{q}_0,\hat{q}) = \min_{(x,u)} \max_{(X,v)} H(t,x,u,X,v,\hat{q}_0,\hat{q})$$

Alternatively, we can also use the dynamic programming approach. Define "the value of the game" $V(\tau,\xi,\Xi)$ by

$$(1.19)\qquad V(\tau,\xi,\Xi) \equiv \min_u \max_v \sum_{i=1}^N \{ \int_\tau^T [h_i(t,x(t),u_1(t),\ldots,u_N(t)) - h_i(t,x^i(t),u_1(t),\ldots,u_{i-1}(t),v_i(t),u_{i+1}(t),\ldots,u_N(t))]dt + g_i(x(T)) - g_i(x^i(T))\}$$

subject to

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^N B_i(t)u_i(t) + f(t), \quad x(\tau) = \xi \in \mathbb{R}^n,$$

$$\dot{x}^i(t) = A(t)x^i(t) + \sum_{\substack{j=1\\ j\neq i}}^N B_j(t)u_j(t) + B_i(t)v_i(t) + f(t),\ x^i(\tau) = \xi^i \in \mathbb{R}^n,\quad 1 \le i \le N,$$

on $[\tau,T] \ni t$, with $\Xi \equiv (\xi^1,\ldots,\xi^N) \in [\mathbb{R}^n]^N$.

If (1.19) is well-defined, under suitable assumptions (cf. [6], §4), we have the Issacs equation

$$(1.20)\qquad \frac{\partial}{\partial\tau}V(\tau,\xi,\Xi) + \min_{u\in \prod_{i=1}^{N}\mathbb{R}^{m_i}}\ \max_{v\in \prod_{i=1}^{N}\mathbb{R}^{m_i}} \{<\nabla_\xi V(\tau,\xi,\Xi), A(\tau)\xi + \sum_{i=1}^{N} B_i(\tau)u_i + f(\tau)>_{\mathbb{R}^n}$$

$$+ \sum_{i=1}^{N} <\nabla_{\xi^i} V(\tau,\xi,\Xi), A(\tau)\xi^i + \sum_{\substack{j=1\\ j\neq i}}^{N} B_j(\tau)u_j + B_i(\tau)v_i + f(\tau)>_{\mathbb{R}^n}$$

$$+ \sum_{i=1}^{N} [h_i(\tau,\xi,u) - h_i(\tau,\xi^i,v^i)] = 0, \qquad (v^i = (u_1,\dots,u_{i-1},v_i,u_{i+1},\dots,u_N))$$

with the terminal conditon

$$(1.21)\qquad V(T,\xi,\Xi) = \sum_{i=1}^{N} [g_i(\xi) - g_i(\xi^i)].$$

This leads further to the Bellman-Hamilton-Jacobi equation

$$(1.22)\qquad \frac{\partial}{\partial t} V(t,x,X) + <\nabla_x V(t,x,X), A(t)x + \sum_{i=1}^{N} B_i(t)\hat{u}_i(t,x,X,\nabla_x V,\nabla_X V) + f(t)>$$

$$+ \sum_{i=1}^{N} <\nabla_{x^i} V(t,x,X), A(t)x^i + \sum_{\substack{j=1\\ j\neq i}}^{N} B_j(t)\hat{u}_j(t,x,X,\nabla_x V,\nabla_X V) +$$

$$+ B_i(t)\hat{v}_i(t,x,X,\nabla_x V,\nabla_X V) + f(t)>$$

$$+ \sum_{i=1}^{N} [h_i(t,x,\hat{u}(t,x,X,\nabla_x V,\nabla_X V)) - h_i(t,x^i,\hat{v}^i(t,x,X,\nabla_x V,\nabla_X V)] = 0,$$

where $\hat{u}(t,x,X,q_0,q)$ and $\hat{v}(t,x,X,q_0,q)$ are "feedback controls" for u and v satisfying

$$\min_{u\in\mathbb{R}^{m_i}} \max_{v\in\mathbb{R}^{m_i}} H(t,x,u,X,v,q_0,q) = \min_{v\in\mathbb{R}^{m_i}} \max_{u\in\mathbb{R}^{m_i}} H(t,x,u,X,v,q_0,q)$$

$$= H(t,x,\hat{u}\,(t,x,X,q_0,q),\ \ X,\hat{v}(t,x,X,q_0,q),q_0,q)$$

for $t \in [0,T]$, $x \in \mathbb{R}^n$, $X \in [\mathbb{R}^n]^N$, $q = (q_1,\ldots,q_N) \in [\mathbb{R}^n]^N$, $q_0 \in \mathbb{R}^n$.

Comparing (1.21), (1.22) with [6] p. 293, (8.2.5), (8.2.6), for example, we see that our B-H-J equation is a single equation (in contrast with a system of N equations), but of $1 + n(N + 1)$ independent variables (in contrast with $1 + n$ variables).

Throughout the above paragraphs, that the min-max point $(\hat{x},\hat{u};\hat{X},\hat{v})$ corresponds to an equilibrium strategy depends on whether the value of (1.9) is 0 or not. This important issue will be addressed in our future papers. For linear quadratic games, a good answer can be found in (4.27) of Theorem 4.7.

§2. Duality Theory

We consider the following inf-sup problem

(P) $$\inf_{x,u}\sup_{X,v} \{J(x,u;X,v) \mid J \text{ as in (1.8)},\ (x,u) \in H^1_n \times U \text{ subject to } (DE) = 0,$$
$$(X,v) \in [H^1_n]^N \times U \text{ subject to } [DE] = 0 \text{ as in (1.10), (1.11)}\} .$$

This constitutes the <u>primal problem</u>. Associated with (P) is the dual problem

(D) $$\sup_{p_0\in L^2_n}\ \inf_{p\in[L^2_n]^N} L\,(p_0,p)\ ,$$

where $p = (p_1,\ldots,p_N)$ and

$$L(p_0,p) = L(p_0,p_1,\dots,p_N) \equiv \inf_{x,u}\ \sup_{X,v}\ \mathcal{L}(p_0,p;x,u;X,v)$$

with the Lagrangian $\mathcal{L}$: $L_n^2 \times [L_n^2]^N \times H_n^1 \times U \times [H_n^1]^N \times U$ defined by

$$\text{(2.1)}\qquad \mathcal{L}(p_0,p;x,u;X,v) \equiv J(x,u;X,v) + \langle p_0, \dot{x} - Ax - \sum_{j=1}^{N} B_j u_j - f\rangle_{L_n^2}$$
$$+ \sum_{i=1}^{N} \langle p_i, \dot{x}^i - Ax^i - \sum_{\substack{j=1\\ j\neq i}}^{N} B_j u_j - B_i v_i - f\rangle_{L_n^2}$$

for x,X satisfying $x(0) = x_0$, $X(0) = X_0 = (x_0,\dots,x_0)$.

From now on we say that (x,u) or (X,v) is <u>feasible</u> if $(x,u) \in H_n^1 \times U$ satisfies (1.10) and $(X,v) \in [H_n^1]^N \times U$ satisfies (1.12). Similarly, (p_0,p) is <u>feasible</u> if $(p_0,p) \in L_n^2 \wedge [L_n^2]^N$.

We are now in a position to state the fundamental theorem in this paper.

<u>Theorem 2.1 (Duality Theorem)</u> Assume that $J(x,u;X,v)$ is convex in (x,u) and concave in (X,v), for all (x,u) and (X,v) satisfying differential constraints, continuous in $H_n^1 \times U \times [H_n^1]^N \times U$ and

$$\text{(A0)}\qquad \inf_{\substack{(x,u)\\ \text{feasible}}}\ \sup_{\substack{(X,v)\\ \text{feasible}}}\ J(x,u;X,v) \equiv \hat{c} < \infty .$$

Then there exists $(\bar{p}_0,\bar{p})$ which is a max-min point for (D) with $L(\bar{p}_0,\bar{p}) = \hat{c}$. Furthermore, if $(\bar{x},\bar{u};\bar{X},\bar{v})$ is a min-max point for (P), then

$$\text{(2.2)}\qquad L(\bar{p}_0,\bar{p}) = \max_{p_0\in L_n^2}\ \min_{p\in[L_n^2]^N}\ L(p_0,p)$$

$$= \max_{p_0 \in L_n^2} \; \min_{p \in [L_n^2]^N} \; \min_{\substack{(x,u) \\ x(0)=x_0}} \; \max_{\substack{(X,v) \\ X(0)=X_0}} \; L(p_0,p;x,u;X,v)$$

$$= \min_{\substack{(x,u) \\ \text{feasible}}} \; \max_{\substack{(X,v) \\ \text{feasible}}} \; J(x,u:X,v)$$

$$= J(\bar{x},\bar{u};\bar{X},\bar{v}).$$

We proceed to prove the theorem.

For any given $(x,u) \in H_n^1 \times U$, let

$$\psi(x,u) \equiv \sup_{\substack{(X,v) \\ \text{feasible}}} J(x,u;X,v), \tag{2.3}$$

and also define

$$\phi(x,u,p) = \sup_{X,v} \{J(x,u;X,v) + \sum_{i=1}^{N} \langle p_i,(DE)_i\rangle \mid X \in [H_n^1]^N, v \in U,\ X(0) = X_0, \tag{2.4}$$
$$p = (p_1,\ldots,p_N) \in [L_n^2]^N\}.$$

By (A0), we know that there exists at least one feasible (x,u) such that

$$\sup_{\substack{(X,v) \\ \text{feasible}}} J(x,u;X,v) = \psi(x,u) < +\infty. \tag{2.5}$$

From now on we need only study $\psi(x,u)$ and $\phi(x,u,p)$ for those (x,u) satisfying (2.5).

Lemma 2.2 (Weak Duality) For any (x,u) satisfying (2.5), the functional $\phi(x,u,p)$ defined above is convex in p and

$$(2.6) \qquad \inf_{p\in[L_n^2]^N} \phi(x,u,p) \geq \psi(x,u)$$

holds.

Proof: Simple verification. □

Lemma 2.4 (Strong Duality) Assume that $J(x,u;X,v)$ is concave in (X,v) for all $(X,v) \in [H_n^1]^N \times U$, $X(0) = X_0$. Then for any $(x,u) \in H_n^1 \times U$, $x(0) = x_0$, we have

$$(2.7) \qquad \inf_{p\in[L_n^2]^N} \phi(x,u,p) = \psi(x,u).$$

Proof: If $\psi(x,u) = +\infty$, then (2.7) holds trivially by Lemma 2.2. So we assume that (2.5) holds. The arguments in [7, p. 846-847] immediately apply. We define two convex sets

$$Y \equiv \{(a,0) \in \mathbb{R} \times [L_n^2]^N \mid a \geq \psi(x,u)\}$$

$$\begin{aligned} Z \equiv \{(a,b) \in \mathbb{R} \times [L_n^2]^N \mid\ & a \leq J(x,u;X,v),\ b = (b_1,\dots,b_N), \\ & b_i = \dot{x}^i - Ax^i - B_i v_i - \sum_{j\neq i} B_j u_j - f, \\ & x^i(0) = x_0,\quad i = 1,\dots,N.\} \end{aligned}$$

Then it is easily checked that $Y \cap$ (interior of $Z) = \phi$ since when $b = 0 \in [L_n^2]^N$,

$$a < J(x,u;X,v) \leq \sup_{\substack{(X,v)\\ \text{feasible}}} J(x,u;X,v)$$

for any $(a,0) \in$ [interior of Z], which is obviously nonempty. So by the separation theorem (see, e.g. [11], p. 38, Theorem 3.3.3), Y and Z can be separated weakly in $\mathbb{R} \times [L_n^2]^N$:

$$(2.8) \qquad r \cdot a_1 + \sum_1^N \langle \bar{q}_i, b_i \rangle_{[L_n^2]^N} \leq r \cdot a_2, \ \forall (a_1,b) \in Z, \ (a_2,0) \in Y,$$

for some $(r,\bar{q}) \in \mathbb{R} \times [L_n^2]^N$. Arguing as in [7], we see that $r > 0$. So r can be normalized to 1. Using $a_1 = J(x,u;X,v)$ and $a_2 = \psi(x,u)$ in (2.8), we get

$$J(x,u;X,v) + \sum_1^N \langle \bar{q}_i, b_i \rangle \leqq \psi(x,u).$$

Therefore

$$\phi(x,u,\bar{q}) \leq \psi(x,u);$$

thus

$$\inf_{p \in [L_n^2]^N} \phi(x,u,p) \leq \phi(x,u,\bar{q}) \leq \psi(x,u).$$

Combining the above with (2.6), we conclude (2.7). □

Remark 2.5 It is well understood in duality theory that the "hyperplane" separating Y and Z will define and attain the optimal dual multipliers [7] (when $\psi(\bar{x},u) < \infty$). □

The arguments for the following lemma are the same as those for Lemmas 2.3 and 2.4, the proofs are therefore omitted.

<u>Lemma 2.6</u> Assume that $J(x,u;X,v)$ is concave with respect to (X,v) and convex with respect to (x,u) for $(X,v) \in [H_n^1]^N \times U$, $(x,u) \in H_n^1 \times U$, $X(0) = X_0$, $x(0) = x_0$. We have

$$\sup_{p_0 \in L_n^2} \inf_{\substack{(x,u) \in H_n^1 \times U \\ x(0)=x_0}} [\psi(x,u) + \langle p_0,(DE)\rangle] = \inf_{\substack{(x,u) \\ \text{feasible}}} [\psi(x,u)]. \qquad \square \tag{2.9}$$

<u>Remark 2.7</u> In (2.4), we introduce N Lagrange multipliers p_i, one for each player. In (2.9), we introduce the joint multiplier p_0 commonly shared by all players. $\square$

<u>Proof of Theorem 2.1</u> From Lemmas 2.4 and 2.7, we conclude that

$$(P) = \inf_{x,u} \sup_{X,u} \{J(x,u;X,v) \mid (x,u) \text{ and } (X,v) \text{ are feasible}\}$$

$$= \inf_{\substack{(x,u) \\ \text{feasible}}} \Big[\sup_{\substack{(X,v) \\ \text{feasible}}} J(x,u;X,v)\Big]$$

$$= \inf_{\substack{(x,u) \\ \text{feasible}}} \psi(x,u)$$

$$= \sup_{p_0 \in L_n^2} \inf_{\substack{(x,u) \in H_n^1 \times U \\ x(0)=x_0}} [\psi(x,u) + \langle p_0,(DE)\rangle] \quad \text{(by Lemma 2.6)}$$

$$= \sup_{p_0 \in L_n^2} \inf_{\substack{(x,u) \in H_n^1 \times U \\ x(0)=x_0}} \inf_{p \in [L_n^2]^N} \sup_{\substack{(X,v) \in [H_n^1]^N \times U \\ X(0)=X_0}} [J(x,u;X,v) + \sum_{i=1}^{N} \langle p_i,(DE)_i \rangle + \langle p_0,(DE) \rangle] \qquad \text{(by Lemma 2.4)}$$

$$= \max_{p_0 \in L_n^2} \min_{p \in [L_n^2]^N} L(p,q) = (D), \qquad \text{(by Remark 2.5).}$$

Hence if $(\bar{x},\bar{u};\bar{X},\bar{v})$ is feasible and solves (P) and if $(\bar{p}_0,\bar{p})$ is feasible and solves (D), we have

$$\hat{c} = J(\bar{x},\bar{u};\bar{X},\bar{v}) = \min_{\substack{(x,u) \\ \text{feasible}}} \max_{\substack{(X,v) \\ \text{feasible}}} J(x,u;X,v)$$

$$= \max_{p_0 \in L_n^2} \min_{p \in [L_n^2]^N} L(p_0,p)$$

$$= L(\bar{p}_0,\bar{p}).$$

So the proof is complete. □

There are still improvements on Theorem 2.1 that could be made, but that would make Theorem 2.1 unduly too general and lengthy, so we choose not to do them here.

§3. Linear Quadratic Problems and Synthesis

From now on throughout the rest of the paper, we consider the linear quadratic problem whose cost functionals are given by

$$(3.1)\qquad J_i(x,u) = \frac{1}{2}\int_0^T [|C_i(t)x(t)-z_i(t)|^2_{\mathbb{R}^{k_i}} + \langle M_i(t)u_i(t),u_i(t)\rangle_{\mathbb{R}^{m_i}}]dt,$$

$$i = 1,\dots,N, \qquad (x,u) \text{ feasible},$$

where we assume that $C_i(t)$ and $M_i(t)$ are matrix-valued functions of appropriate sizes and smoothness, $z_i(t)$ is a vector-valued function. Furthermore, $M_i(t)$ induces a linear operator $M_i\colon L^2_{m_i} \to L^2_{m_i}$ which is positive definite:

$$(3.2)\qquad \langle M_i u_i, u_i\rangle_{L^2_{m_i}} \ge \nu_0 \|u_i\|^2_{L^2_{m_i}}, \quad 1 \le i \le N,$$

for some $\nu_0 > 0$.

The main objective of this section is to give a formal derivation of the adjoint equations and the Riccati equation from the dual formulation. Later on in §4 we will see that under certain sufficient conditions these procedures can be justified by Theorems 2.1 and 4.6.

We use the definition of $J(x,u;X,v)$ as in (1.8). For any feasible (p_0,p), the Lagrangian L is

$$(3.3)\qquad L(p_0,p;x,u;X,v) = J(x,u;X,v) + \langle p_0,(DE)\rangle_{L^2_n} + \sum_{i=1}^N \langle p_i,(DE)_i\rangle_{L^2_n}$$

$$= \sum_{i=1}^{N} [J_i(x,u_1,\ldots,u_N) - J_i(x^i,u_1,\ldots,u_{i-1},v_i,u_{i+1},\ldots,u_N)]$$

$$+ \langle p_0, \dot{x} - Ax - \sum_{i=1}^{N} B_i u_i - f\rangle_{L_n^2}$$

$$+ \sum_{i=1}^{N} \langle p_i, \dot{x}^i - Ax^i - \sum_{j\neq 1} B_j u_j - B_i v_i - f\rangle_{L_n^2} \quad .$$

We first study $\max\limits_{\substack{(X,v)\\ X(0)=X_0}} L(p_0,p;x,u;X,v)$. Assume that for given $p_0,p,x,u,$ the maximum is attained at $(\hat{X},\hat{v})$. By a simple variational analysis on x^i, we have, necessarily,

$$(3.4) \qquad -\langle C_i^*(C_i\hat{x}^i - z_i), y^i\rangle_{L_n^2} + \langle p_i, \dot{y}^i - Ay^i\rangle_{L_n^2} = 0 \quad , \quad (C^* = \text{adjoint of } C),$$

for all $y^i \in H_n^1$, $y^i(0) = 0$, $1 \le i \le N$.

From variational analysis, we also have

$$(3.5) \qquad p_i \in H_n^1, \quad p_i(T) = 0,$$

and

$$-\langle C_i^*(C_i\hat{x}^i - z_i) + \dot{p}_i + A^* p_i, y^i\rangle_{L_n^2} = 0; \qquad 1 \le i \le N.$$

Hence

$$(3.6)\qquad \dot{p}_i = -A^* p_i - C_i^*(C_i\hat{x}^i - z_i).$$

Similar variational analysis on v_i gives

$$-\langle M_i\hat{v}_i, w_i\rangle_{L^2_{m_i}} - \langle p_i, B_i w_i\rangle_{L^2_n} = 0, \qquad \forall w_i \in L^2_{m_i},$$

or,

$$(3.7)\qquad \hat{v}_i = -M_i^{-1}B_i^* p_i, \qquad 1 \le i \le N.$$

We now consider $\min\limits_{\substack{(x,u)\\ x(0)=x_0}} L(p_0,p;x,u;\hat{X},\hat{v})$. Assume that the minimum is attained at $(\hat{x},\hat{u})$. By the same reasoning as above, we get

$$(3.8)\qquad p_0 \in H^1_n, \quad p_0(T) = 0,$$

$$(3.9)\qquad \dot{p}_0 = -A^* p_0 + \sum_{i=1}^{N} C_i^*(C_i\hat{x} - z_i),$$

$$(3.10)\qquad \hat{u}_i = M_i^{-1}B_i^*(p_0 + \sum_{\substack{j=1\\ j\ne i}}^{N} p_j) = M_i^{-1}B_i^*(p_0 + p_s - p_i), \qquad p_s \equiv \sum_{j=1}^{N} p_j.$$

Let $L(p_0,p)$ be as defined in §2. If the problem $\max\limits_{p_0}\min\limits_{p} L(p_0,p)$ attains its max-min at $(\hat{p}_0,\hat{p})$, then $\hat{p}_0$ and $\hat{p}$ satisfy (3.8), (3.9) and (3.5), (3.6). Therefore we obtain $\hat{X},\hat{v},\hat{x},\hat{u},\hat{p}_0,\hat{p}$ as the solution to the following two point boundary value problem:

<u>Theorem 3.1</u> Let $\hat{X},\hat{v},\hat{x},\hat{u},\hat{p}_0$ and $\hat{p}$ satisfy

$$L(\hat{p}_0,\hat{p}) = \max_{p_0\in L_n^2}\ \min_{p\in[L_n^2]^N} L(p_0,p)$$

$$= \max_{p_0}\ \min_{p} \mathcal{L}(p_0,p;\hat{x},\hat{u};\hat{X},\hat{v})$$

$$= \max_{p_0}\ \min_{p}\ \min_{\substack{x,u\\x(0)=x_0}}\ \max_{\substack{X,u\\X(0)=X_0}} \mathcal{L}(p_0,p;x,u;X,v)$$

$$= J(\hat{x},\hat{u};\hat{X},\hat{v})$$

$$= \min_{\substack{x,u\\x(0)=x_0}}\ \max_{\substack{X,v\\X(0)=X_0}} J(x,u;X,v).$$

Then $\hat{x},\hat{X} = (\hat{x}^1,\ldots,\hat{x}^N)$, $\hat{p}_0,\hat{p} = (\hat{p}_1,\ldots,\hat{p}_N)$ are coupled through:

$$(3.11)\quad \frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{x}^1 \\ \vdots \\ \hat{x}^N \\ \hat{p}_0 \\ \hat{p}_1 \\ \vdots \\ \hat{p}_N \end{bmatrix} = \begin{bmatrix} A & 0 & \cdots & 0 & S & S_1 & \cdots & S_N \\ 0 & A & & 0 & S_1 & S_{11} & \cdots & S_{1N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & & \\ 0 & 0 & & A & S_N & S_{N1} & \cdots & S_{NN} \\ \sum_{i=1}^{N} C_i^* C_i & 0 & \cdots & 0 & -A^* & 0 & \cdots & 0 \\ 0 & -C_1^* C_1 & & 0 & 0 & -A^* & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & -C_N^* C_N & 0 & & & -A^* \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x}^1 \\ \vdots \\ \hat{x}^N \\ \hat{p}_0 \\ \hat{p}_1 \\ \vdots \\ \hat{p}^N \end{bmatrix} + \begin{bmatrix} f \\ f \\ \vdots \\ f \\ \sum_{i=1}^{N} C_i^* z_i \\ C_1^* z_1 \\ \vdots \\ C_N^* z_N \end{bmatrix}$$

$$\hat{x}(0) = \hat{x}^1(0) = \ldots. = \hat{x}^N(0) = x_0,$$

$$\hat{p}_0(T) = \hat{p}_1(T) = \ldots = \hat{p}_N(T) = 0\ ,$$

and $\hat{u}, \hat{v}$ satisfy

$$\hat{u}_i = M_i^{-1} B_i^* (\hat{p}_0 + \hat{p}_s - \hat{p}_i),$$

$$\hat{v}_i = -M_i^{-1} B_i^* \hat{p}_i,$$

where in (3.11),

$$(3.12)\qquad S \equiv \sum_{j=1}^{N} B_j M_j^{-1} B_j^*,$$

$$S_i \equiv \sum_{j \neq i} B_j M_j^{-1} B_j^*,$$

$$S_{ik} \equiv S - (1 - \delta_{ik}) B_i M_i^{-1} B_i^* - B_k M_k^{-1} B_k^*, \quad (\delta_{ik} = \text{Kronecker's delta}). \qquad \square$$

Decoupling can be achieved by assuming the feedback affine relation

$$(3.13)\qquad \begin{bmatrix} \hat{p}_0 \\ \hat{p} \end{bmatrix} = \mathbb{P} \begin{bmatrix} \hat{x} \\ \hat{X} \end{bmatrix} + \mathbb{r}.$$

Let us write

$$\mathbb{S} \equiv \begin{bmatrix} S & S_1 & \cdots\cdots & S_N \\ S_1 & S_{11} & & S_{1N} \\ \vdots & \vdots & & \vdots \\ S_N & S_{N1} & & S_{NN} \end{bmatrix}$$

$$\mathbb{A} \equiv \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \ddots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{bmatrix}_{[n\times(N+1)]\times[n\times(N+1)]}$$

$$\mathbb{C} \equiv \begin{bmatrix} \sum_{i=1}^{N} C_i^* C_i & 0 & \cdots & 0 \\ 0 & -C_1^* C_1 & \ddots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & -C_N^* C_N \end{bmatrix}$$

$$-\mathbb{A}^* = \begin{bmatrix} -A^* & 0 & \cdots & 0 \\ 0 & \cdots -A^* & \cdots & 0 \\ 0 & \cdots & 0 & -A^* \end{bmatrix}$$

which denote, respectively, the first, second, third and fourth quadrant of blocks of matrices in the big matrix in (3.11). From (3.11) and (3.13), using the above notations, we get the Riccati equation

$$(3.14) \quad \begin{cases} \dot{\mathbb{P}} + \mathbb{P}\mathbb{A} + \mathbb{A}^*\mathbb{P} + \mathbb{P}\mathbb{S}\mathbb{P} - \mathbb{C} = 0, \\ \mathbb{P}(T) = 0, \end{cases}$$

for $\mathbb{P}$. We also have

$$(3.15) \quad \begin{cases} \dot{\mathbb{r}} + (\mathbb{P}\mathbb{S} + \mathbb{A}^*)\mathbb{r} + \mathbb{P}f - \zeta = 0, \\ \mathbb{r}(T) = 0, \end{cases}$$

where

$$\zeta \equiv \begin{bmatrix} -\sum_{i=1}^{N} C_i^* z_i \\ C_1^* z_1 \\ \vdots \\ C_N^* z_N \end{bmatrix}.$$

The reader may compare the Riccati equation (3.14) from our dual approach with that in [9, (4.30)] obtained form the primal approach or that in [6, p. 312, (8.5.23)].

§4. The Dual Max-Min Problem

We study the dual problem in this section. This will become the basis of the finite element computations in §5.

Henceforth, for simplicity, we denote the operators $C_i^*C_i$ and $\sum_{i=1}^{N} C_i^*C_i$ (induced by the matrices $C_i^*(t)C_i(t)$ and $\sum_{i=1}^{N} C_i^*(t)C_i(t)$) in L_n^2 as $\mathbb{C}_i (1 \le i \le N)$ and $\mathbb{C}_0$, respectively.

We will need several assumptions as we proceed. First, we assume

(A1) each operator $\mathbb{C}_i (1 \le i \le N)$ is strictly positive definite in L_n^2.

From (3.6), we get

$$\hat{x}^i = -\mathbb{C}_i^{-1}(\dot{p}_i + A^* p_i - C_i^* z_i). \tag{4.1}$$

By (A1), $\mathbb{C}_0$ is also strictly positive definite. By (3.9), we get

$$\hat{x} = \mathbb{C}_0^{-1}(\dot{p}_0 + A^* p_0 + \sum_{i=1}^{N} C_i^* z_i). \tag{4.2}$$

We now substitute (4.1), (4.2), (3.7) and (3.10) into (3.3). Integrating by parts with respect to p_0 and $p_i (1 \le i \le N)$ once, using the end conditions (3.5) and (3.8) and simplifying, one obtains

$$L(p_0,p) = \mathcal{L}(p_0,p;\hat{x},\hat{u};\hat{X},\hat{v}) \tag{4.3}$$

$$= -\frac{1}{2}\langle \dot{p}_0 + A^* p_0, \mathbb{C}_0^{-1}(\dot{p}_0 + A^* p_0)\rangle + \frac{1}{2}\sum_{i=1}^{N}\langle \dot{p}_i + A^* p_i, \mathbb{C}_i^{-1}(\dot{p}_i + A^* p_i)\rangle$$

$$-\frac{1}{2}<p_0+p_s,S(p_0+p_s)>+<p_0+p_s,\sum_{i=1}^{N} B_iM_i^{-1}B_i^*p_i>-<\dot{p}_0+A^*p_0,\mathbb{C}_0^{-1}\sum_{i=1}^{N} C_i^*z_i>$$

$$-\sum_{i=1}^{N}<\dot{p}_i+A^*p_i,\mathbb{C}_i^{-1}C_i^*z_i> - <p_0+p_s,f> - <p_0(0)+p_s(0),x_0>$$

$$-\frac{1}{2}<\mathbb{C}_0^{-1}(\sum_{j=1}^{N} C_j^*z_j),\sum_{j=1}^{N} C_j^*z_j>+\frac{1}{2}\|z\|^2$$

$$\equiv \sum_{i=1}^{10} \mathrm{II}_i,$$

where $\|z\|^2 \equiv \sum_{i=1}^{N} \|z_i\|^2_{L^2_{k_i}}$, and p_s, S are defined as in §3.

We are now faced with the problem of $\max_{p_0} \min_{p} L(p_0,p)$. It is easy to see that $L(p_0,p)$ is strictly concave in p_0 for any given p. However, for any given p_0, $L(p_0,p)$ is not necessarily convex in p because of the negative sign in front of II_3. To circumvent this, we will need the following import assumption:

(A2) The positive definite operators $\mathbb{C}_i^{-1}(1 \le i \le N)$ in L_n^2 are large enough so that

$$(4.4)\qquad \frac{1}{2}\sum_{i=1}^{N}<\dot{p}_i+A^*p_i,\mathbb{C}_i^{-1}(\dot{p}_i+A^*p_i)>-\frac{1}{2}<p_s,Sp_s>+<p_s,\sum_{i=1}^{N} B_iM_i^{-1}B_i^*p_i>$$

$$\ge \nu_1 \sum_{i=1}^{N} \|\dot{p}_i\|^2 ,$$

for all $p_i \in H^1_{0n} \equiv \{q|q,\dot{q} \in L^2_n, q(T) = 0\}$, for some $\nu_1 > 0$.

We remark that even if $\mathbb{C}_i^{-1}$, $1 \le i \le N$, are not large enough, the above assumption can still be valid provided that T is chosen sufficiently small, because in this case, the first positive definite quadratic form in (4.4) will

have a larger coercivity coefficient to bound the L^2-norm, when the interval $[0,T]$ is small. This agrees with the assumption that $t_1 - t_0$ be sufficiently small in [9, p. 114, line 15].

Another special case wherein (A2) holds without requiring $\mathbb{C}_i^{-1}$ $1 \leq i \leq N$ to be large is when

$$N = 2, \quad U_1 = U_2$$

$$B_1 M_1^{-1} B_1^* = B_2 M_2^{-1} B_2^* \equiv B, \quad \text{for some } B \geq 0.$$

It is easily seen that now

$$(4.4) = \frac{1}{2} \sum_{i=1}^{2} < \dot{p}_i + A^* p_i, \mathbb{C}_i^{-1} (\dot{p}_i + a^* p_i) > - \frac{1}{2} \cdot 2 < p_s, Bp_s > + < p_s, Bp_s >$$

$$= \frac{1}{2} \sum_{i=1}^{2} < \dot{p}_i + A^* p_i, \mathbb{C}_i^{-1} (p_i + A^* p_i) > ,$$

so (A2) holds.

<u>Remark 4.1</u> We believe that if (4.4) is not a positive semi-definite quadratic form, then $J(x,u;X,v)$ is not convex in (x,u) (J is always concave in (X,v) for any given u), thus hindering the existence or uniqueness of the equilibrium strategy. This is still being investigated. □

Because the quadratic form (4.4) is symmetric, using the end condition $p_i(T) = 0$ $(1 \leq i \leq N)$ and the Poincaré inequality, we see that for $p^1 = (p_1^1, \ldots, p_N^1)$, $p^2 = (p_1^2, \ldots, p_N^2) \in [H_{0n}^1]^N$, the bilinear form

$$(4.5) \qquad \beta(p^1, p^2) \equiv \frac{1}{2} \sum_{i=1}^{N} <\dot{p}_i^1 + A^* p_i^1, \mathbb{C}_i^{-1}(\dot{p}_i^2 + A^* p_i^2)> - \frac{1}{2}<p_s^1, Sp_s^2>$$

$$+ \frac{1}{2}<p_s^1, \sum_1^N B_i M_i^{-1} B_i^* p_i^2> + \frac{1}{2}<p_s^2, \sum_1^N B_i M_i^{-1} B_i^* p_i^1>$$

defines an equivalent inner product in $[H_{0n}^1]^N$.

<u>Lemma 4.2</u> Under (A1) and (A2), for each given p_0, $L(p_0,p)$ is strictly convex in p and for each given p, $L(p_0,p)$ is strictly concave in p_0.

<u>Proof</u>: For each given $\bar{p}_0 \in H^1_{0n}$, we can write $L(\bar{p}_0,p)$ as

$$L(\bar{p}_0,p) = \beta(p,p) + \text{linear terms in } p + \text{constant terms (depending on } \bar{p}_0 \text{ and } z_i).$$

Since β forms an equivalent inner product in $[H^1_{0n}]^N$, we conclude that $L(\bar{p}_0,p)$ is strictly convex in p.

The second assertion is already clear. □

For each given p_0, $L(p_0,p)$ is strictly convex, continuous and coercive in p (i.e., $L(p_0,p) \to +\infty$ as $\|p\|_{[H^1_{0n}]^N} \to +\infty$). Therefore

$$\min_{p\in[H^1_{0n}]^N} L(p_0,p) = L(p_0,\hat{p}(p_0)) \tag{4.6}$$

is uniquely attained at $\hat{p}(p_0)$, depending on p_0.

From a straightforward variational analysis (or the Euler-Lagrange equations), we see that $\hat{p}(p_0)$ satisfies

$$(4.7)\quad \begin{cases} \frac{d}{dt}\mathbb{C}_i^{-1}(\dot{\hat{p}}_i + A^*\hat{p}_i) - A\mathbb{C}_i^{-1}(\dot{\hat{p}}_i + A^*\hat{p}_i) + S(p_0 + \hat{p}_s) - \sum_1^N B_j M_j^{-1} B_j^* \hat{p}_j \\ \quad - B_i M_i^{-1} B_i^*(p_0 + \hat{p}_s) + A\mathbb{C}_i^{-1} C_i^* z_i - \frac{d}{dt}(\mathbb{C}_i^{-1} C_i^* z_i) + f = 0, \\ \hat{p}_i(T) = 0, \\ \mathbb{C}_i^{-1}(0)[\dot{\hat{p}}_i(0) + A^*(0)\hat{p}_i(0)] = -x_0 + \mathbb{C}_i^{-1}(0)C_i^*(0)z_i(0); \quad 1 \le i \le N, \end{cases}$$

where it is assumed that $\mathbb{C}_i^{-1}C_i^*z_i$ $(1 \le i \le N)$ are sufficiently smooth so that $\{\mathbb{C}_i^{-1}(0)C_i^*(0)z_i(0)\}_1^N$ exist.

Now, consider $\bar{L}(p_0) \equiv L(p_0,\hat{p}(p_0))$ as a functional of p_0. It is easy to verify that $\bar{L}(p_0)$ is concave with respect to p_0. In fact, we have

<u>Lemma 4.3</u> $\bar{L}(p_0)$ is <u>strictly</u> concave with respect to p_0.

<u>Proof</u>: For any $\theta \in [0,1]$ and any $p_0^1, p_0^2 \in H_{0n}^1$, we have

$$(4.8) \qquad \bar{L}(\theta p_0^1 + (1-\theta)p_0^2) = \min_{p\in[H_{0n}^1]^N} L(\theta p_0^1 + (1-\theta)p_0^2, p)$$

$$= \min_{p\in[H_{0n}^1]^N} \{-\varepsilon\|[\theta\dot{p}_0^1 + (1-\theta)\dot{p}_0^2] + A^*[\theta p_0^1 + (1-\theta)p_0^2]\|^2_{L_n^2}$$

$$+ [L(\theta p_0^1 + (1-\theta)p_0^2,p) + \varepsilon\|[\theta\dot{p}_0^1 + (1-\theta)\dot{p}_0^2] + A^*[\theta p_0^1 + (1-\theta)p_0^2]\|^2_{L_n^2}]\}$$

where in the above, ε is chosen sufficiently small so that $-\frac{1}{2}\mathbb{C}_0^1 + \varepsilon I$ is still strictly negative definite. Continuing from (4.8), we get

$$(4.8) = -\varepsilon\|(\tfrac{d}{dt} + A^*)[\theta p_0^1 + (1-\theta)p_0^2]\|^2_{L_n^2} + \min_{p\in[H_{0n}^1]^N} \{L(\theta p_0^1 + (1-\theta)p_0^2,p) + \varepsilon\|(\tfrac{d}{dt}+A^*)\cdot$$

$$[\theta p_0^1 + (1-\theta)p_0^2]\|^2_{L_n^2}\}$$

$$\ge -\varepsilon\|(\tfrac{d}{dt}+A^*)[\theta p_0^1 + (1-\theta)p_0^2]\|^2_{L_n^2}$$

$$+ \min_{p\in[H_{0n}^1]^N} \{\theta[L(p_0^1,p) + \varepsilon\|\dot{p}_0^1 + A^*p_0^1\|^2] + (1-\theta)[L(p_0^2,p) + \varepsilon\|\dot{p}_0^2 + A^*p_0^2\|^2]\},$$

because the parenthesized term is concave and because $-\frac{1}{2}\Sigma_0^{-1} + \varepsilon I$ is negative definite.

(continuing from the above)

$$(4.9) \qquad \geq -\varepsilon\|(\tfrac{d}{dt}+A^*)[\theta p_0^1+(1-\theta)p_0^2]\|^2_{L_n^2} + \theta \min_{p\in[H_{0n}^1]^N} \{L(p_0^1,p)+\varepsilon\|\dot p_0^1+A^*p_0^1\|^2\}$$

$$+ (1-\theta) \min_{p\in[H_{0n}^1]^N} \{L(p_0^2,p)+\varepsilon\|\dot p_0^2+A^*p_0^2\|^2\}.$$

If $p_0^1 \neq p_0^2$ and $\theta \neq 0,1$, then

$$-\varepsilon\|(\tfrac{d}{dt}+A^*)[\theta p_0^1+(1-\theta)p_0^2]\|^2 + \theta\varepsilon\|\dot p_0^1+A^*p_0^1\|^2+(1-\theta)\varepsilon\|\dot p_0^2+A^*p_0^2\|^2 > 0,$$

so (4.8) and (4.9) give

$$\bar L(\theta p_0^1+(1-\theta)p_0^2) > \theta \min_{p\in[H_{0n}^1]^N} L(p_0^1,p)+(1-\theta) \min_{p\in[H_{0n}^1]^N} L(p_0^2,p)$$

$$= \theta\bar L(p_0^1) + (1-\theta)\bar L(p_0^2),$$

proving strict concavity. □

We proceed to study $\max_{p_0\in H_{0n}^1} \bar L(p_0)$.

<u>Lemma 4.4</u> Under (A1) and (A2), $\bar L(p_0)$ is (negatively) coercive with respect to p_0, i.e.,

$$\bar L(p_0) \to -\infty \quad \text{as} \quad \|p_0\|_{H_{0n}^1} \to +\infty.$$

<u>Proof</u>: Because $0 \in [H^1_{0n}]^N$, we have

$$(4.10)\qquad \bar{L}(p_0) = \min_{p\in[H^1_{0n}]^N} L(p_0,p) \le L(p_0,0)$$

$$= -\frac{1}{2}<\dot{p}_0 + A^* p_0, \mathbb{C}_0^{-1}(\dot{p}_0 + A^* p_0)> - \frac{1}{2}<p_0, Sp_0> - <p_0, f>_{L^2_n} - <p_0(0), x_0>_{\mathbb{R}^n}$$

$$- \frac{1}{2}<\mathbb{C}_0^{-1}(\sum_1^N C_j^* z_j), \sum_1^N C_j^* z_j> + \frac{1}{2}\|z\|^2 - <\dot{p}_0 + A^* p_0, \mathbb{C}_0^{-1}\sum_{i=1}^N C_i^* z_i>$$

We use

$$|<p_0, f>_{L^2_n}| \le \frac{\varepsilon}{2}\|p_0\|^2_{L^2_n} + \frac{1}{2\varepsilon}\|f\|^2_{L^2_n} \le \frac{\varepsilon}{2}K<\dot{p}_0 + A^* p_0, \mathbb{C}_0^{-1}(\dot{p}_0 + A^* p_0)> + \frac{1}{2\varepsilon}\|f\|^2,$$

$$|<\dot{p}_0 + A^* p_0, \mathbb{C}_0^{-1}\sum_{i=1}^N C_i^* z_i>| \le \frac{\varepsilon}{2}<\dot{p}_0 + A^* p_0, \mathbb{C}_0^{-1}(\dot{p}_0 + A^* p_0)> + \frac{1}{2\varepsilon}<\sum_{i=1}^N C_i^* z_i, \mathbb{C}_0^{-1}\sum_{i=1}^N C_i^* z_i$$

$$|<p_0(0), x_0>_{\mathbb{R}^n}| \le \frac{\varepsilon}{2}\|p_0(0)\|^2_{\mathbb{R}^n} + \frac{1}{2\varepsilon}\|x_0\|^2_{\mathbb{R}^n} \le \frac{\varepsilon}{2}K<\dot{p}_0 + A^* p_0, \mathbb{C}_0^{-1}(\dot{p}_0 + A^* p_0)>$$

$$+ \frac{1}{2\varepsilon}\|x_0\|^2$$

in (4.10); in the above the constant $K > 0$ depends on $\mathbb{C}_0^{-1}$ only. Choose ε sufficiently small. One sees that

$$\bar{L}(p_0) \le L(p_0,0) \le (-\frac{1}{2} + \varepsilon K + \frac{\varepsilon}{2})<\dot{p}_0 + A^* p_0, \mathbb{C}_0^{-1}(\dot{p}_0 + A^* p_0)> - \frac{1}{2}<p_0, Sp_0>$$

$$+ [\frac{1}{2\varepsilon}\|x_0\|^2 + \frac{1}{2\varepsilon}\|f\|^2 + (\frac{1}{2\varepsilon} - 1)<\mathbb{C}_0^{-1}(\sum_1^N C_j^* z_j), \sum_1^N C_j^* z_j> + \frac{1}{2}\|z\|^2],$$

the right hand side tends to $-\infty$ as $\|p_0\|_{H^1_{0n}} \to +\infty$. □

The first main theorem in this section is

<u>Theorem 4.5</u> <u>(Dual Saddle Point Theorem)</u> Under (A1), (A2), the max-min problem $\max_{p_0} \min_{p} L(p_0,p)$ has a unique solution $(\hat{p}_0,\hat{p})$. Furthermore,

$$(4.11) \qquad \max_{p_0 \in H^1_{0n}} \; \min_{p \in [H^1_{0n}]^N} L(p_0,p) = \min_{p \in [H^1_{0n}]^N} \; \max_{p_0 \in H^1_{0n}} L(p_0,p).$$

<u>Proof</u>: We use the standard saddle point argument [5], except that we replace the compactness condition by coercivity.

For each p_0, there exists a unique $\hat{p}(p_0)$ minimizing $L(p_0,p)$ with respect to p as in (4.6).

By Lemmas 4.3 and 4.4, $\max_{p_0 \in H^1_{0n}} \bar{L}(p_0) = L(p_0,\hat{p}(p_0))$ also has a unique minimizer $\hat{p}_0$. Hence $\max_{p_0} \min_{p} L(p_0,p)$ has a unique solution $(\hat{p}_0,\hat{p})$ (with $\hat{p} = \hat{p}(\hat{p}_0)$):

$$(4.12) \qquad \bar{L}(\hat{p}_0) = \max_{p_0 \in H^1_{0n}} \bar{L}(p_0,\hat{p}(p_0)) = \max_{p_0 \in H^1_{0n}} \; \min_{p \in [H^1_{0n}]^N} L(p_0,p) = \min_{p \in [H^1_{0n}]^N} L(\hat{p}_0,p).$$

For any $p_0 \in H^1_{0n}$, $p \in [H^1_{0n}]^N$ and $\theta \in (0,1)$, we have

$$\begin{aligned} L((1-\theta)\hat{p}_0 + \theta p_0,p) &\geq (1-\theta)L(\hat{p}_0,p) + \theta L(p_0,p) \\ &\geq (1-\theta)\bar{L}(\hat{p}_0) + \theta L(p_0,p). \end{aligned}$$

In particular, we choose $p = \hat{p}((1-\theta)\hat{p}_0 + \theta p_0)$. From the above we get

$$\bar{L}(\hat{p}_0) \geq \bar{L}((1-\theta)\hat{p}_0 + \theta p_0) \geq (1-\theta)\bar{L}(\hat{p}_0) + \theta L(p_0, \hat{p}((1-\theta)\hat{p}_0 + \theta p_0)).$$

Hence

$$\bar{L}(\hat{p}_0) \geq L(p_0, \hat{p}((1-\theta)\hat{p}_0 + \theta p_0)).$$

Noting that $\hat{p}((1-\theta)\hat{p}_0 + \theta p_0)$ is continuous with respect to θ, one lets θ tend to $0+$ and gets

$$\bar{L}(\hat{p}_0) \geq L(p_0, \hat{p}(\hat{p}_0)), \qquad \forall p_0 \in H^1_{0n}.$$

On the other hand, from (4.12),

$$\bar{L}(\hat{p}_0) \leq L(\hat{p}_0, q), \qquad \forall q \in [H^1_{0n}]^N.$$

Therefore we conclude

$$L(p_0, \hat{p}(\hat{p}_0)) = L(p_0, \hat{p}) \leq \bar{L}(\hat{p}_0) = L(\hat{p}_0, \hat{p}) \leq L(\hat{p}_0, p), \forall p, p_0.$$

Hence (4.11) is proved. □

So far, our derivation of the dual problem is only formal because we have not yet verified the assumptions in Theorem 2.1 that $J(x,u,X,v)$ is convex in (x,u) and concave in (X,v) and that $\inf \sup J(x,u,X,v)$ is attainable. These questions are answered in the following theorem.

<u>Theorem 4.6</u> (Primal-Dual Equivalence Theorem)

Assume that $C_i(t)$, $z_i(t)$, $1 \le i \le N$, $f(t)$ and $\mathcal{C}_0^{-1}$, $\mathcal{C}_i^{-1}$, $1 \le i \le N$, are sufficiently smooth (as functions and operators, respectively). Under assumptions (A1) and (A2), for the linear quadratic differential game (0.1) and (3.1), let $J(x,u;X,v)$ be defined as in (1.8). Then

i) $J(x,u;X,v)$ is convex in (x,u) and strictly concave in (X,v);

ii) there exist unique $(\hat{x},\hat{u})$ and $(\hat{X},\hat{v})$ such that

$$(4.13)\qquad \inf_{\substack{(x,u)\\ \text{feasible}}} \sup_{\substack{(X,v)\\ \text{feasible}}} J(x,u;X,v) = \min_{\substack{(x,u)\\ \text{feasible}}} \max_{\substack{(X,v)\\ \text{feasible}}} J(x,u;X,v)$$

$$= J(\hat{x},\hat{u};\hat{X},\hat{v}) < \infty \quad ;$$

iii)

$$(4.14)\qquad \min_{(x,u)} \max_{(X,v)} J(x,u;X,v) = \max_{(X,v)} \min_{(x,u)} J(x,u;X,v)$$

iv)

$$(4.15)\qquad L(\hat{p}_0,\hat{p}) = \max_{p_0 \in L_n^2} \min_{p\in[L_n^2]^N} L(p_0,p)$$

$$= \max_{p_0 \in L_n^2} \min_{p\in[L_n^2]^N} \min_{\substack{(x,u)\\ x(0)=x_0}} \max_{\substack{(X,v)\\ X(0)=X_0}} L(p_0,p;x,u;X,v).$$

$$= \min_{\substack{(x,u)\\ \text{feasible}}} \max_{\substack{(X,v)\\ \text{feasible}}} J(x,u;X,v).$$

v) The (second) dual of the (first) dual problem (namely, (D)), obtained by regarding $\dot{p}_i - \frac{d}{df} p_i = 0$ $(1 \le i \le N)$ as constraints in L, recoveres to the primal problem (P).

Proof: The proof is based upon the "reflexivity" argument that "the dual of the dual is primal".

By Theorem 4.5, $L(p_0,p)$ attain its unique saddle point at $(\hat{p}_0,\hat{p})$.

In finding the saddle point of $L(p_0,p)$, we regard $\dot{p}_i - \frac{d}{dt} p_i = 0$, $0 \le i \le N$, as constraints and introduce Lagrange multipliers λ_0, $\lambda=(\lambda_1,\ldots,\lambda_N)$ and consider

$$\inf_{\lambda_0 \, L_n^2} \; \sup_{\lambda \, L_n^{2\,N}} \; \sup_{\substack{p_0,\dot{p}_0 \\ p_0(T)=0}} \; \inf_{\substack{p,\dot{p} \\ p(T)=0}} \; \mathcal{I}(p_0,\dot{p}_0,p,\dot{p};\lambda_0,\lambda)$$

where

$$\mathcal{I}(p_0,\dot{p}_0,p,\dot{p};\lambda_0,\lambda) \equiv \left[L(p_0,\dot{p}_0,p,\dot{p}) + \langle\lambda_0,\dot{p}_0 - \frac{d}{dt}p_0\rangle + \sum_{i=1}^{N} \langle\lambda_i,p_i - \frac{d}{dt}p_i\rangle\right], \tag{4.16}$$

and $L(p_0,\dot{p}_0,p,\dot{p})$ is the same as that in (4.3) except that we now regard p_0 and $\dot{p}_0$ as unrelated.

Define

$$I(\lambda_0,\lambda) \equiv \sup_{\substack{p_0,\dot{p}_0 \\ p_0(T)=0}} \; \inf_{\substack{p,\dot{p} \\ p(T)=0}} \; \mathcal{I}(p_0,\dot{p}_0,p,\dot{p};\lambda_0,\lambda) \tag{4.17}$$

We now apply (the proof of) Theorem 2.1 to $L(p_0,p)$, subject to constraints $\dot{p}_i - \frac{d}{dt} p_i = 0$, $p_i(T)=0$, $0 \le i \le N$. It is easy to see that all the assumptions of Theorem 2.1 are satisfied by $L(p_0,p)$, since by (A1) and (A2), $L(p_0,p)$ is strictly convex in p and strictly concave in p_0. So we have a unique $(\hat{\lambda}_0,\hat{\lambda}) \in L_n^2 \times [L_n^2]^N$ such that

$$I(\hat{\lambda}_0, \hat{\lambda}) = \min_{\lambda_0 \in L_n^2} \quad \max_{\lambda \in [L_n^2]^N} \quad I(\lambda_0, \lambda)$$

$$= \min_{\lambda_0} \max_{\lambda} \max_{\substack{p_0, \dot{p}_0 \\ p_0(T)=0}} \min_{\substack{p, \dot{p} \\ p(T)=0}} I(p_0, \dot{p}_0, p, \dot{p}; \lambda_0, \lambda)$$

$$= \max_{\substack{p_0 \\ p_0(T)=0}} \min_{\substack{p \\ p(T)=0}} L(p_0, p) = L(\hat{p}_0, \hat{p}).$$

On the other hand, from (4.16) and (4.17), by variational analysis on the p_0, $\dot{p}_0$, p, $\dot{p}$ variables, we have, necessarily, that $\lambda_0 \in H_n^1$, $\lambda \in [H_n^1]^N$ and

$$\lambda_0 - \mathcal{E}_0^{-1}(\dot{p}_0 + A^* p_0 + \sum_{i=1}^{N} C_j^* z_j) = 0, \tag{4.18}$$

$$\dot{\lambda}_0 - [A\mathcal{E}_0^{-1}(\dot{p}_0 + A^* p_0 + \sum_{j=1}^{N} C_j^* z_j) - S(p_0 + p_s) + \sum_{j=1}^{N} B_j M_j^{-1} B_j^* p_j + f] = 0, \tag{4.19}$$

$$\lambda_i + \mathcal{E}_i^{-1}(\dot{p}_i + A^* p_i - C_i^* z_i) = 0 \tag{4.20}$$

$$\dot{\lambda}_i + [\, A\mathcal{E}_i^{-1}(\dot{p}_i + A^* p_i - \sum_{j=1}^{N} C_j^* z_j) + S(p_0 + p_s) - \sum_{j=1}^{N} B_j M_j^{-1} B_j^* p_j - B_i M_i^{-1} B_i^* p_i - f] = 0, \tag{4.21}$$

$$1 \le i \le N.$$

In the above, p_0, $\dot{p}_0$, p, $\dot{p}$ depend on λ_0, λ. Now define $\eta = (\eta_1, \ldots, \eta_N)$ and $\zeta = (\zeta_1, \ldots, \zeta_N)$ by

$$(4.22) \qquad \eta_i \equiv M_i^{-1} B_i^* (p_0 + p_s - p_i) \qquad , \qquad 1 \le i \le N,$$

$$(4.23) \qquad \zeta_i \equiv -M_i^{-1} B_i^* p_i \qquad , \qquad 1 \le i \le N.$$

From (3.12),(4.18), (4.19) and (4.22), we see that λ_0 satisfies

$$(4.24) \qquad \dot{\lambda}_0 = A\lambda_0 + S(p_0 + p_s) - \sum_{j=1}^{N} B_j M_j^{-1} B_j^* p_j + f$$

$$= A\lambda_0 + \sum_{j=1}^{N} B_j \left[M_j^{-1} B_j^* (p_0 + p_s - p_j) \right] + f$$

$$= A\lambda_0 + \sum_{j=1}^{N} B_j \eta_j + f.$$

Similarly, from (3.12), (4.20) - (4.23), we get

$$(4.25) \qquad \dot{\lambda}_i = A\lambda_i + \sum_{j \ne i} B_j \eta_j + B_i \zeta_i + f.$$

The initial conditions satisfied by λ_0, $\lambda_i (1 \le i \le N)$ are just

$$(4.26) \qquad \lambda_0(0) = x_0, \qquad \lambda_i(0) = x_0 \qquad , \quad 1 \le i \le N.$$

This can be easily verified (e.g., by comparing (4.18) with (4.7.4)).

Substituting (4.18), (4.20), (4.22) and (4.23) into $L(p_0,p)$, we get $I(\lambda_0,\lambda) \equiv \tilde{I}(\lambda_0,\eta;\lambda,\zeta)$, which is convex in (λ_0,η) and concave in (λ,ζ). But this $\tilde{I}(\lambda_0,\eta;\lambda,\zeta)$ is just $J(x,u;X,v)$ through identifying $(\lambda_0,\eta,\lambda,\zeta)$ with $(x,u;X,v)$, subject to (4.24) - (4.26), i.e., subject to (1.10)=0 and (1.11)=0 $(0 \le i \le N)$

$J(x,u;X,v)$ is convex in (x,u) and concave in (X,v) because $J(x,u;X,v) = \tilde{I}(\lambda_0,\eta;\lambda,\zeta)$, which is the dual of $L(p_0,p)$ which is convex in p_0 and concave in p. The fact that $J(x,u;X,v)$ is strictly concave in (X,v) for any given (x,u) can be verified directly from J itself.

The min-max and max-min in (4.14) are exchangeable because of (4.11) in Theorem 4.5.

$\square$

<u>Theorem 4.7</u> (Existence and Uniqueness of Equilibrium Strategy for N-person Linear-Quadratic Differential Games)

Assume that (A1) and (A2) hold. Then the unique saddle point $(\hat{x},\hat{u};\hat{X},\hat{v})$ of (4.14) satisfies the property that $\hat{u} = \hat{v}$ and $\hat{x}^i = \hat{x}$ on $[0,T]$, where $\hat{x}^i$ is the i-th component of $\hat{X}$.

Thus

$$J(\hat{x},\hat{u};\hat{X},\hat{v}) = \min_{(x,u)} \max_{(X,v)} J(x,u;X,v) = \max_{(X,v)} \min_{(x,u)} J(x,u;X,v) = 0, \tag{4.27}$$

so $\hat{u}$ is the unique equilibrium strategy for the N-person differential game.

<u>Proof:</u> By (4.14), the saddle point property for J is uniquely satisfied by $(\hat{x},\hat{u};\hat{X},\hat{v})$, so we have

$$\min_{\substack{(x,u)\\ \text{feasible}}} \max_{\substack{(X,v)\\ \text{feasible}}} J(x,u;X,v) = \max_{(X,v)} J(\hat{x},\hat{u};X,v).$$

Since the RHS above is uniquely attained by $(\hat{X},\hat{v})$ ($\hat{X}$ depends on both $\hat{u}$ and $\hat{v}$), we see that v is uniquely characterized by

$$\partial_{v_i} J(\hat{x},\hat{u};X,v)\Big|_{v=\hat{v}} = -\partial_{v_i} J_i(x^i,\hat{u}_1,\ldots,\hat{u}_{i-1},v_i,\hat{u}_{i+1},\ldots\hat{u}_N)\Big|_{v_i=\hat{v}_i} = 0, \tag{4.28}$$

where ∂_{v_i} denotes the Fréchet derivative with respect to v_i.

Similarly, we have

$$\min_{(x,u)} \max_{(X,v)} J(x,u;X,v) = \min_{(x,u)} J(x,u;\tilde{X}(u,\hat{v}),\hat{v})$$

where in the RHS above $\tilde{X}(u,\hat{v}) = (\tilde{x}^1(u,\hat{v}),\ldots,\tilde{x}^N(u,\hat{v}))$ depends on u and $\hat{v}$ as follows:

$$\begin{cases} \dot{\tilde{x}}^i = A\tilde{x}^i + \sum_{j\neq i} B_j u_j + B_i\hat{v}_i + f \\ \tilde{x}^i(0) = x_0. \end{cases}$$

Thus $\hat{u}$ is uniquely characterized by

$$(4.29)\qquad \partial_{u_i} J(x,u;\tilde{X}(u,\hat{v}),\hat{v}) = \sum_{j=1}^{N} \partial_{u_i}\Big[J_j(x,u,\ldots,u_N) - J_j(\tilde{x}^j(u,\hat{v}),u_1,\ldots,u_{j-1},\hat{v}_j,u_{j+1},\ldots u_N)\Big]$$

$$= 0 \quad \text{at} \quad u = \hat{u}, \quad \text{for} \quad i = 1,\ldots,N.$$

Therefore (4.29) gives

$$(4.30)\qquad \partial_{u_i} J(x,\hat{u}^i;\tilde{X}(\hat{u}^i,\hat{v}),\hat{v}) \Big|_{u_i=\hat{u}_i} = 0,$$

where

$$\hat{u}^i = (\hat{u}_1,\ldots,\hat{u}_{i-1},u_i,\hat{u}_{i+1},\ldots,\hat{u}_N), \quad \text{for} \quad i = 1,\ldots,N.$$

But, evaluating the RHS of (4.29) with $u = \hat{u}^i$, at $u_i = \hat{u}_i$, we find that

$$\partial_{u_i}\Big[J_j(x,\hat{u}_1,\ldots\hat{u}_{i-1},u_i,\hat{u}_{i+1},\ldots,\hat{u}_N) - J_j(\tilde{x}^j(\hat{u}^i,\hat{v}),\hat{u}_1,\ldots\hat{u}_{i-1},u_i,\hat{u}_{i+1},\ldots,\hat{u}_{j-1},\hat{v}_j\,\hat{u}_{j+1},\ldots,\hat{u}_N)\Big]\Big|_{u_i=\hat{u}_i} = 0,$$

if $j \neq i$.

So (4.30) is reduced to

$$(4.31)\quad \partial_{u_i}\Big[J_j(x,\hat{u}_1,\ldots\hat{u}_{i-1},u_i,\hat{u}_{i+1},\ldots,\hat{u}_N) - J_i(\tilde{x}^j(\hat{u}^i,\hat{v}),\hat{u}_1,\ldots,\hat{u}_{i-1},\hat{v}_i,\hat{u}_{i+1},\ldots,\hat{u}_N)\Big]\Big|_{u_i}$$

$$= \partial_{u_i} J_i(x,\hat{u}_1,\ldots,\hat{u}_{i-1},u_i,\hat{u}_{i+1},\ldots\hat{u}_N)\Big|_{u_i=\hat{u}_i} = 0,$$

because the second term in the above bracket is just a constant.

Comparing (4.28) with (4.31), we see that $\hat{u}_i$ and $\hat{v}_i$, $i = 1,\ldots,N$, satisfy the very same equations, whose solutions are unique. Hence $\hat{u} = \hat{v}$ is proved.

Because $\hat{u} = \hat{v}$, we conclude immediately that $\hat{x}^i = \hat{x}, \forall i = 1,\ldots,N$ and that the saddle point value (4.27) is 0. So $\hat{u}$ is an equilibrium strategy. □

Remark 4.8 The above theorem says that, under (A1) and (A2), any N-person non zero-sum linear quadratic differential game is, indeed, a 2N-person zero-sum game, with N authentic players represented by u_i, $1 \le i \le N$, and N fictitious players represented by v_i, $1 \le i \le N$. □

Remark 4.9 If, at the outset, we consider

(4.32) $$\min_{(x,u)} \max_{(X,v)} \{ J(x,u;X,v) \equiv \sum_{i=1}^{N} [J_i(x,u_1,\ldots,u_N) - J_i(x^i,u_1,\ldots,u_{i-1}, v_i,u_{i+1},\ldots u_N)] \mid (x,u) \text{ and } (X,v) = (x^1,\ldots,x^N,v) \text{ satisfy (4.33), (4.34) below} \}.$$

(4.33) $$\begin{cases} \dot{x} = Ax + \sum_i B_i u_i + f & \text{on } [0,T], \\ x(0) = x_0, \end{cases}$$

(4.34) $$\begin{cases} \dot{x}^i = Ax^i + \sum_{j \ne i} B_j u_j + B_i v_i + f & \text{on } [0,T], \\ x^i(0) = x_0^i, & 1 \leqq i \leqq N \end{cases}$$

Note that in (4.34.2), x_0^i $(1 \leqq i \leqq N)$ need not be equal to x_0 in (4.33.2). Then using duality, we will arrive at the same $L(p_0,p)$ as given in (4.3), except that II_8 is now replaced by

$$\mathrm{II}_8' \equiv -\langle p_0(0),x_0 \rangle - \sum_{i=1}^{N} \langle p_i(0),x_0^i \rangle.$$

Since the validity of assumptions (A1) and (A2) is not affected by $\mathbb{T}'_8$, we see that all the theorems in this section, except Corollary 4.7, remain valid for problem (4.32). The result $\hat{u} = \hat{v}$ still holds for problem (4.32). But, now $\hat{x}^i \neq \hat{x}$ in general, so the saddle point value of (4.32) is not equal to 0 in general. □

<u>Remark 4.10</u> For linear-quadratic N-person games, under (A1) and (A2), the Hamiltonian (1.18) and the Bellman-Hamilton-Jacobi equation must be at a saddle point (instead of just min-max) for all t or $\tau \in [0,T]$. □

§5 Global Existence and Uniqueness of Solutions for the Riccati Equation

The system of Riccati equations [8,(4.30)] in Lukes and Russell's approach has been known to have only local existence and uniqueness of solutions. However, under our approach, we can prove that our Riccati equation has global existence and uniqueness of solutions. The proof is an extension of the control theory case, cf. e.g. [12, pp.197-205], to our equation.

<u>Theorem 5.1</u> Under assumptions (A1) and (A2), the Riccati equation

$$(5.1)\qquad \begin{cases} \dot{\mathbb{P}} + \mathbb{P}\,A + A^{*}\mathbb{P} + \mathbb{P}\mathbb{S}\mathbb{P} - \mathbb{C} = 0 & , \text{ on } [0,T], \\ \mathbb{P}(T) = 0 \end{cases}$$

as given in (3.14) has a unique solution $\mathbb{P}$ on $[0,T]$.

<u>Proof:</u> Define

$$J_i(x,u;\xi_0;t_0,t_1) \equiv \int_{t_0}^{t_1} \left[|C_i(t)x(t)|^2 + \langle M_i(t)u_i(t),u_i(t)\rangle\right]dt, \quad 1 \leqq i \leqq N$$

subject to

$$(5.2)\qquad \begin{cases} \frac{d}{dt}\,x(t) = A(t)x(t) + \sum\limits_{i=1}^{N} B_i(t)\,u_i(t) & , \quad t \in [t_0,t_1] \\ x(t_0) = \xi_0 & . \end{cases}$$

and

$$\bar{J}_i(x^i,u_1,\ldots,u_{i-1},v_i,u_{i+1},\ldots,u_N;\xi_i;t_0,t_1) \equiv \int_{t_0}^{t_1} \left[|C_i(t)x^i(t)|^2 + \langle M_i(t)v_i(t),v_i(t)\rangle\right]dt, \quad 1 \leqq i \leqq N,$$

subject to

$$(5.3)\qquad \begin{cases} \frac{d}{dt}\,x^i(t) = A(t)x^i(t) + \sum\limits_{j\neq i} B_j(t)u_j(t) + B_i(t)v_i(t) & , \ t \in [t_0,t_1] \\ x^i(t_0) = \xi_i, & 1 \leqq i \leqq N. \end{cases}$$

Further, let

$$(5.4)\quad J(x,u;X,v;\xi_0,\xi_1,\dots,\xi_N;t_0,t_1) \equiv \sum_{i=1}^{N} \Big[J_i(x,u;\xi_0;\ t_0,t_1) - \bar{J}_i(x^i,u_1,\dots,u_{i-1}, v_i,u_{i+1},\dots,u_N;\xi_i;t_0,t_1)\Big],$$

subject to (5.2) and (5.3).

<u>Lemma 5.2</u> Assume (A1), (A2). Let $(\hat{x},\hat{u}:\hat{X},\hat{v})$ satisfy (5.2) and (5.3) with $t_0 = 0$, $t_1 = T$, $\xi_0 = \xi_1 = \dots = \xi_N = x_0$. Let $q_0, q_1,\dots,q_N$ be the solution on $[0,T]$ of

$$(5.5)\qquad \dot{q}_0 = -A^*q_0 + \sum_{i=1}^{N} \mathbb{C}_i\hat{x}, \qquad q_0(T) = 0,$$

$$(5.6)\qquad \dot{q}_i = -A^*q_i - \mathbb{C}_i\hat{x}^i, \qquad q_i(T) = 0, \qquad 1 \leqq i \leqq N.$$

Then $(\hat{x},\hat{u};\hat{X},\hat{v})$ is the unique saddle point for $\min\limits_{(x,u)}\ \max\limits_{(X,v)}\ J(x,u;X,v,x_0,X_0;0,T)$

if and only if

$$(5.7)\qquad \hat{u}_i = M_i^{-1}B_i^*(q_0 + \sum_{j\neq i} q_j) \quad , \quad 1 \leqq i \leqq N$$

$$(5.8)\qquad \hat{v}_i = -M_i^{-1}B_i^*q_i \quad , \quad 1 \leqq i \leqq N.$$

<u>Proof of Lemma 5.2</u> By Theorem 4.7, $\min\limits_{(x,u)}\ \max\limits_{(X,v)}\ J(x,u;X,v;x_0,X_0;0,T)$ has a unique saddle point. If $(\hat{x},\hat{u};\hat{X},\hat{v})$ is this saddle point, it is characterized by

$$(5.9)\qquad J(\hat{x},\hat{u};\hat{X},\hat{v};x_0,X_0;0,T) \leqq J(\hat{x}+\varepsilon\tilde{x};\hat{u}+\varepsilon\tilde{u};\hat{X}+\varepsilon\bar{X},\hat{v},x_0X_0;0,T), \qquad \forall\varepsilon \in \mathbb{R},$$

$$(5.10)\qquad J(\hat{x},\hat{u};\hat{X},\hat{v};x_0,X_0;0,T) \geqq J(\hat{x},\hat{u};\hat{X}+\varepsilon\tilde{X},\hat{v}+\varepsilon\tilde{v};x_0,X_0;0,T), \qquad \forall\varepsilon \in \mathbb{R},$$

where $(\tilde{x},\tilde{u})$ and $(\tilde{X},\tilde{v})$ and $\bar{X} \in (\bar{x}^1,\bar{x}^2,\dots,\bar{x}^N)$ satisfy

$$(5.11)\qquad \begin{cases} \dot{\tilde{x}} = A\tilde{x} + \sum_i B_i\tilde{u}_i \qquad \text{on } [0,T], \\ \tilde{x}(0) = 0, \end{cases}$$

$$(5.12)\quad \begin{cases} \dot{\tilde{x}}^i = A\tilde{x}^i + \sum\limits_{j\neq i} B_j\hat{u}_j + B_i\tilde{v}_i & \text{on } [0,T] \\ \tilde{x}^i(0) = 0, & 1 \leqq i \leqq N. \end{cases}$$

$$(5.13)\quad \begin{cases} \dot{\bar{x}}^i = A\bar{x}^i + \sum\limits_{j\neq i} B_j\tilde{u}_j & \text{on } [0,T], \\ \bar{x}^i(0) = 0, & 1 \leqq i \leqq N. \end{cases}$$

Note that in the RHS of (5.9) $\hat{X} + \varepsilon\bar{X}$ appears because it is also dependent on $\hat{u} + \varepsilon\tilde{u}$.

From (5.9), we get

$$\frac{d}{d\varepsilon} J(\hat{x} + \varepsilon\tilde{x},\ \hat{u} + \varepsilon\tilde{u};\ \hat{X} + \varepsilon\bar{X},\ \hat{v};\ x_0,\ X_0;\ 0,\ T)\Big|_{\varepsilon=0} = 0,$$

which is

$$(5.14)\qquad 2\sum_{i=1}^{N}\int_0^T \Big[\langle C_i(t)\hat{x}(t), C_i(t)\tilde{x}(t)\rangle + \langle M_i(t)\hat{u}_i(t), \tilde{u}_i(t)\rangle - \langle C_i(t)\hat{x}(t), C_i(t)\bar{x}^i(t)\rangle\Big]\,dt = 0$$

From (5.5), (5.6), (5.11), (5.12) and (5.13), we have

$$0 = \langle\tilde{x}(T), q_0(T)\rangle + \sum_i \langle\bar{x}^i(T), q_i(T)\rangle$$

$$= \langle\tilde{x}(0), q_0(0)\rangle + \sum_i \langle\bar{x}^i(0), q_i(0)\rangle + \int_0^T \frac{d}{dt}\Big[\langle\tilde{x}(t), q_0(t)\rangle + \sum_i \langle\bar{x}^i(t), q_i(t)\rangle\Big]\,dt$$

$$= \int_0^T \Big[\langle\dot{\tilde{x}}(t), q_0(t)\rangle + \langle\tilde{x}(t), \dot{q}_0(t)\rangle + \sum_i \langle\dot{\bar{x}}^i(t), q_i(t)\rangle + \sum_i \langle\bar{x}^i(t), \dot{q}_i(t)\rangle\Big]\,dt$$

$$= \int_0^T \Big[\langle A(t)\tilde{x}(t) + \sum_i B_i(t)\tilde{u}_i(t), q_0(t)\rangle + \langle\tilde{x}(t), -A^*(t)q_0(t) + \sum_i \mathbb{C}_i(t)\hat{x}(t)\rangle$$

$$+ \sum_i \langle A(t)\bar{x}^i(t) + \sum_{j\neq i} B_j(t)\tilde{u}_j(t), q_i(t)\rangle + \sum_i \langle\bar{x}^i(t), -A^*(t)q_i(t) - \mathbb{C}_i(t)\hat{x}^i(t)\rangle\Big]\,dt$$

$$(5.15) \quad = \sum_i \int_0^T [< C_i(t)\hat{x}(t), C_i(t)\tilde{x}(t)> + <M_i(t)\hat{u}_i(t),\tilde{u}_i(t)> - <C_i(t)\hat{x}^i(t), C_i(t)\bar{x}^i(t)>]dt$$

$$+ \sum_i \int_0^T [-<M_i(t)\hat{u}_i(t),\tilde{u}_i(t)> + <B_i^*(t)q_0(t),\tilde{u}_i(t)> + \sum_{j\neq i} <B_i^*(t)q_j(t),\tilde{u}_i(t)>] \, dt.$$

Comparing (5.15) with (5.14), we see that (5.9) holds if and only if

$$\int_0^T [-<M_i(t)\hat{u}_i(t),\tilde{u}_i(t)> +< B_i^*(t)q_0(t),\tilde{u}_i(t)> + \sum_{j\neq i} <B_i^*(t)q_j(t),\tilde{u}_i(t)>]dt = 0$$

for all $\tilde{u}_i \in U_i$, $i = 1,2,\ldots,N$. This gives

$$-M_i\hat{u}_i + B_i^*q_0 + \sum_{j\neq i} B_iq_j = 0, \qquad 1 \leq i \leq N,$$

which are just (5.7).

We can obtain (5.8) in a similar manner. The proof of Lemma 5.2 is complete.

The proof of Lemma 5.2 indicates that with appropriate simple adaptation, the arguments given in [12,pp.197-205] are immediately applicable to our proof. As in [12,p.199,(2.16)], analogously, we now claim that we have

$$(5.16) \qquad q_0(\tau)\hat{x}(\tau) + \sum_{i=1}^N q_i(\tau)\hat{x}^i(\tau) = \min_{(x,u)} \max_{(X,v)} J(x,u;X,v;\hat{x}(\tau),\hat{X}(\tau);\tau,T), \quad \tau \in [0,$$

where $(\hat{x},\hat{u};\hat{X},\hat{v})$ solves the min-max problem, on the RHS above with (arbitrary) initial condition $(\hat{x}(\tau),\hat{X}(\tau))$ for (x,X) at the beginning time τ.

Because f and ζ in (3.14) are 0, the solution γ of (3.14) is also 0 on $[0,T]$. Thus, by (3.12),

$$(5.17) \qquad \begin{bmatrix} q_0(t) \\ q(t) \end{bmatrix} = \mathbb{P}(t) \begin{bmatrix} \hat{x}(t) \\ \hat{X}(t) \end{bmatrix}, \qquad t \in [0,T], \text{ if } \mathbb{P} \text{ exists.}$$

From (5.16) and (5.17), we get

$$(5.18) \qquad < \mathbb{P}(\tau) \begin{bmatrix} \hat{x}(\tau) \\ \hat{X}(\tau) \end{bmatrix}, \begin{bmatrix} \hat{x}(\tau) \\ \hat{X}(\tau) \end{bmatrix} > = \min_{(x,u)} \max_{(X,v)} J(x,u;X,v;\hat{x}(\tau),\hat{X}(\tau);\tau,T),$$

whenever $\mathbb{P}$ exists on $[\tau,T]$.

The nonlinearity in the Riccati equation (3.13) satisfies the local Lipschitz condition. So, by the Picard local existence and uniqueness theorem, the solution $\mathbb{P}(t)$ of (3.14) exists at least on a half open interval $(\tau', T]$, for some $\tau' < T$. Assume the contrary that $\mathbb{P}$ does not exist globally on $[0,T]$. Then there is at least one $\tau' \in [0,T)$ such that

$$\lim_{t \downarrow \tau'} ||P(t)|| = \infty .$$

This means that there exists at least one $(x_0, x_0^1, x_0^2, \ldots, x_0^N) \in [\mathbb{R}^n]^{N+1}$ such that

(5.19)

$$\text{(5.19)} \qquad \lim_{t \downarrow \tau'} \left| < \mathbb{P}(t) \begin{bmatrix} x_0 \\ x_0^1 \\ \vdots \\ x_0^N \end{bmatrix}, \begin{bmatrix} x_0 \\ x_0^1 \\ \vdots \\ x_0^N \end{bmatrix} > \right| = \infty .$$

But, if we choose $t_0 = \tau'$, $\xi_0 = x_0$, $\xi_i = x_0^i$ $(1 \leq i \leq N)$ in (5.2), (5.3) and apply (5.18) and Remark 4.8, we see that

$$\lim_{t \downarrow \tau'} < \mathbb{P}(t) \begin{bmatrix} x_0 \\ x_0^1 \\ \vdots \\ x_0^N \end{bmatrix}, \begin{bmatrix} x_0 \\ x_0^1 \\ \vdots \\ x_0^N \end{bmatrix} > = \min_{(x,u)} \max_{(X,u)} J(x,u;X,v;x_0,(x_0^1,\ldots,x_0^N);\tau',T)$$

$$= \text{a finite number},$$

contradicting (5.19).

Therefore $\mathbb{P}$ exists uniquely on $[0,T]$.

∎

§6. The Dual Variational Problem and Finite Element Approximations

Let $F: H_1 \times H_2 \to \mathbb{R}$ be a real-valued Fréchet differentiable mapping from a product Hilbert space $H_1 \times H_2$ into $\mathbb{R}$. Assume that $F(x,y)$ is strictly convex in x (for each y) and strictly concave in y (for each x), and that $(\hat{x},\hat{y})$ is the unique saddle point of F satisfying

$$\min_{x\in H_1} \max_{y\in H_2} F(x,y) = \max_{y\in H_2} \min_{x\in H_1} F(x,y).$$

Then it can be easily shown that $(\hat{x},\hat{y})$ is uniquely characterized by

$$(6.1) \qquad \partial_x F(x,\hat{y})\big|_{x=\hat{x}} = 0,$$

$$(6.2) \qquad \partial_y F(\hat{x},y)\big|_{y=\hat{y}} = 0.$$

We now apply the above property to $L(p_0,p)$. It is easy to see from the theory in §4 that all of the assumptions above are satisfied. Therefore $(\hat{p}_0,\hat{p})$, the unique solution of $\max\limits_{p_0} \min\limits_{p} L(p_0,p)$ in $H^1_{0n} \times [H^1_{0n}]^N$, is characterized by $\partial_{p_0} L(\hat{p}_0,\hat{p}) = 0$, $\partial_p L(\hat{p}_0,\hat{p}) = 0$.

From (4.3), by a simple calculation, we get

$$(6.3) \qquad \partial_{p_0} L(\hat{p}_0,\hat{p})\cdot r = -<\dot{\hat{p}}_0 + A^*\hat{p}_0, \mathbb{C}_0^{-1}(\dot{r} + A^* r)> - <\hat{p}_0 + \hat{p}_s, Sr> + <r, \sum_1^N B_i M_i^{-1} B^* \hat{p}_i>$$

$$- <\dot{r} + A^* r, \mathbb{C}_0^{-1} \sum_1^N C_i^* z_i> - <r,f> - <r(0),x_0> = 0, \ \forall r \in H^1_{0n},$$

$$(6.4) \qquad \partial_p L(\hat{p}_0,\hat{p})\cdot s = \sum_1^N <\dot{\hat{p}}_i + A^*\hat{p}_i, \mathbb{C}_i^{-1}(\dot{s}_i + A^* s_i)> - <\hat{p}_0 + \hat{p}_s, S \sum_{i=1}^N s_i>$$

$$+ <\hat{p}_0 + \hat{p}_s, \sum_1^N B_i M_i^{-1} B_i^* s_i> + <\Sigma s_i, \sum_1^N B_i M_i^{-1} B_i^* \hat{p}_i>$$

$$- \sum_1^N <\dot{s}_i + A^* s_i, \mathbb{C}_i^{-1} C_i^* z_i> - <\sum_1^N s_i, f> - <\sum_1^N s_i(0), x_0> = 0,$$

$$\forall s = (s_1,\ldots,s_N) \in [H^1_{0n}]^N.$$

The above two relations induce a bilinear form on $H^1_{0n} \times [H^1_{0n}]^N$: for r^1, $r^2 \in H^1_{0n}$ and $s^1 = (s^1_1,\ldots,s^1_N)$, $s^2 = (s^2_1,\ldots,s^2_N) \in [H^1_{0n}]^N$,

$$(6.5) \qquad a\left(\begin{bmatrix} r^1 \\ s^1 \end{bmatrix}, \begin{bmatrix} r^2 \\ s^2 \end{bmatrix}\right) \equiv -<\dot r^1 + A^* r^1, \mathbb{C}_0^{-1}(\dot r^2 + A^* r^2)>$$

$$- <r^1 + \sum_{j=1}^N s^1_j, Sr^2> + <r^2, \sum_1^N B_i M_i^{-1} B_i^* s^1_i>$$

$$+ \sum_1^N <\dot s^1_i + A^* s^1_i, \mathbb{C}_i^{-1}(\dot s^2_i + A^* s^2_i)> - <r^1 + \sum_{j=1}^N s^1_j, S \sum_{j=1}^N s^2_j>$$

$$+ <r^1 + \sum_{i=1}^N s^1_i, \sum_{i=1}^N B_i M_i^{-1} B_i^* s^2_i> + <\sum_{i=1}^N s^2_i, \sum_{i=1}^N B_i M_i^{-1} B_i^* s^1_i>,$$

and a linear form θ: for $r \in H^1_{0n}$ and $s = (s_1,\ldots,s_N) \in H^1_{0n}$,

$$(6.6) \qquad \theta\left(\begin{bmatrix} r \\ s \end{bmatrix}\right) \equiv <r + \sum_1^N s_j, f> + <r(0) + \sum_1^N s_j(0), x_0> + <\dot r + A^* r, \mathbb{C}_0^{-1} \sum_1^N C_i^* z_i>$$

$$+ \sum_1^N <\dot s_i + A^* s_i, \mathbb{C}_i^{-1} C_i^* z_i>$$

Thus (6.3) and (6.4) are equivalent to

$$(6.7) \qquad a\left(\begin{bmatrix} \hat p_0 \\ \hat p \end{bmatrix}, \begin{bmatrix} r \\ s \end{bmatrix}\right) = \theta\left(\begin{bmatrix} r \\ s \end{bmatrix}\right), \qquad \forall (r,s) \in H^1_{0n} \times [H^1_{0n}]^N.$$

We are now in a position to compute $(\hat{p}_0,\hat{p})$ by the finite element method. As in [1], we say that $S_h^2 \subset H_\ell^{t_2}(0,T)$ is a <u>(t_1,t_2)-system</u> (t_1,t_2 are non-negative integers) if for all $v \in H_\ell^k(0,T)$, there exists $v_h \in S_h$ such that

$$\|v - v_h\|_{H_\ell^\eta} \le Kh^\mu \|v\|_{H_\ell^{\mu+\eta}}, \quad \forall 0 \le \eta \le \min(k,t_2),\ \eta \in \mathbb{N}, \tag{6.8}$$

where $\mu = \min(t_1 - \eta, k - \eta)$ and $K > 0$ is independent of h and v.

Let $S_h \subset H_{0n}^1$ be a $(\tau,1)$-system. We consider

$$\max_{p_0 \in S_h} \min_{p \in [S_h]^N} L(p_0,p). \tag{6.9}$$

It is easy to see that under (A1), (A2), there exists a unique saddle point $(\hat{p}_{0h},\hat{p}_h) \in S_h \times [S_h]^N$ such that

$$L(\hat{p}_{0h},\hat{p}_h) = \max_{p_0 \in S_h} \min_{p \in [S_h]^N} L(p_0,p).$$

This point $(\hat{p}_{0h},\hat{p}_h)$ is characterized as the solution to the variational equation

$$a\left(\begin{bmatrix}\hat{p}_{0h}\\ \hat{p}_h\end{bmatrix}, \begin{bmatrix}r_h\\ s_h\end{bmatrix}\right) = \theta\left(\begin{bmatrix}r_h\\ s_h\end{bmatrix}\right), \quad \forall (r_h,s_h) \in S_h \times [S_h]^N. \tag{6.10}$$

If $\{\varphi^i\}_{i=1}^J$, $\{\psi^i\}_{i=1}^{NJ}$ are basis for S_h, $[S_h]^N$, respectively, then (6.10) is a matrix equation $\bar{M}_h\bar{\gamma}_h = \bar{\theta}_h$, where

$$[\bar{M}_h]_{ij} = a\left(\begin{bmatrix}\psi^i \\ \varphi^i\end{bmatrix}, \begin{bmatrix}\psi^j \\ \varphi^j\end{bmatrix}\right) \quad , \qquad 1 \le i,\ j \le (N+1)J,$$

$$(\bar{\theta}_h)_j = \theta\left(\begin{bmatrix}\psi^j \\ \varphi^j\end{bmatrix}\right) \quad , \qquad 1 \le j \le (N+1)J.$$

More specifically,

$$\bar{M}_h = \left[\begin{array}{l|l}
-\langle \dot{\psi}^i + A^*\psi^i, \mathbb{C}_0^{-1}(\dot{\psi}^j + A^*\psi^j)\rangle & -\langle \sum_{k=1}^{N} \varphi_k^i, S\psi^j\rangle + \langle \psi^j, \sum_{k=1}^{N} B_k M_k^{-1} B_k^* \varphi_k^i\rangle \\
\quad - \langle \psi^i, S\psi^j\rangle & \\
\hline
-\langle \psi^i, S \sum_{k=1}^{N} \varphi_k^j\rangle + \langle \psi^i, \sum_{k=1}^{N} B_k M_k^{-1} B_k^* \varphi_k^j\rangle & \sum_{k=1}^{N} \langle \dot{\varphi}_k^i + A^*\varphi^i, \mathbb{C}_i^{-1}(\dot{\varphi}_k^j + A^*\varphi_k^j)\rangle - \\
 & \quad - \langle \sum_{k=1}^{N} \varphi_k^i, S \sum_{k=1}^{N} \varphi_k^j\rangle \\
 & + \langle \sum_{k=1}^{N} \varphi_k^i, \sum_{k=1}^{N} B_k M_k^{-1} B_k^* \varphi_k^j\rangle + \\
 & + \langle \sum_{k=1}^{N} \varphi_k^j, \sum_{k=1}^{N} B_k M_k^{-1} B_k^* \varphi_k^i\rangle
\end{array}\right]$$

$$\bar{\theta}_h = \left[\begin{array}{l}
\langle \psi^j, f\rangle + \langle \psi^j(0), x_0\rangle \\
+ \langle \dot{\psi}^j + A^*\psi^j, \mathbb{C}_0^{-1} \sum_{k=1}^{N} C_k^* z_k\rangle \\
\hline
\langle \sum_{k=1}^{N} \varphi_k^j, f\rangle + \langle \sum_{k=1}^{N} \varphi_k^j(0), x_0\rangle \\
+ \sum_{k=1}^{N} \langle \dot{\varphi}_k^j + A^*\varphi_k^j, \mathbb{C}_k^{-1} C_k^* z_k\rangle
\end{array}\right]$$

Note that $\bar{M}_h$ is symmetric but non-positive definite.

Numerical analysis for general quadratic saddle point problems seems to be difficult. To make the above computations amenable to standard finite element error analysis, once again, we need two more assumptions:

(A3) the bilinear form a satisfies

$$\inf_{\left\|\begin{bmatrix} r^2 \\ s^2 \end{bmatrix}\right\| = 1} \; \sup_{\left\|\begin{bmatrix} r^1 \\ s^1 \end{bmatrix}\right\| = 1} \left| a\left(\begin{bmatrix} r^1 \\ s^1 \end{bmatrix}, \begin{bmatrix} r^2 \\ s^2 \end{bmatrix} \right) \right| > 0;$$

and

(A4) the spaces $\{S_h\}_h$ satisfy

$$\inf_{\left\|\begin{bmatrix} r_h^2 \\ s_h^2 \end{bmatrix}\right\| = 1} \; \sup_{\left\|\begin{bmatrix} r_h^1 \\ s_h^1 \end{bmatrix}\right\| = 1} \left| a\left(\begin{bmatrix} r_h^1 \\ s_h^1 \end{bmatrix}, \begin{bmatrix} r_h^2 \\ s_h^2 \end{bmatrix} \right) \right| \equiv \gamma_h > \gamma > 0,$$

for some $\gamma > 0, \forall h > 0$.

The fact that the above two assumptions are realistic can be seen from the following

Proposition 6.1 If $\mathbb{C}_i^{-1}$, $i = 0,1,\ldots,N$, as positive definite operators, are comparatively larger than S and $B_i M_i^{-1} B_i^*$, $i = 1,\ldots,N$, then (A3) and (A4) are valid.

Proof For any given $(r^2,s^2) \in H_{0n}^1 \times [H_{0n}^1]^N$ (or, $(r_h^2,s_h^2) \in S_h \times [S_h]^N$), we have

$$(6.11) \quad \sup_{\left\|\begin{bmatrix} r^1 \\ s^1 \end{bmatrix}\right\| = 1} \left| a\left(\begin{bmatrix} r^1 \\ s^1 \end{bmatrix}, \begin{bmatrix} r^2 \\ s^2 \end{bmatrix} \right) \right| \geq \left| a\left(\begin{bmatrix} -r^2 \\ s^2 \end{bmatrix}, \begin{bmatrix} r^2 \\ s^2 \end{bmatrix} \right) \right|$$

$$\geq [<\dot{r}^2 + A^*r^2, \mathbb{C}_0^{-1}(\dot{r}^2 + A^*r^2)> + \sum_1^N <\dot{s}_i^2 + A^*s_i^2, \mathbb{C}_i^{-1}(\dot{s}_i^2 + A^*s_i^2)> + <r^2, Sr^2>]$$

$$- [< \sum_{j=1}^N s_j^2, Sr^2> - <r^2, \sum_{i=1}^N B_i M_i^{-1} B_i^* s_i^2> + <-r^2 + \sum_{j=1}^N s_j^2, S \sum_{j=1}^N s_j^2>$$

$$+ <r^2 - \sum_{i=1}^N s_i^2, \sum_{i=1}^N B_i M_i^{-1} B_i^* s_i^2> - < \sum_{i=1}^N s_i^2, \sum_{i=1}^N B_i M_i^{-1} B_i^* s_i^2>].$$

If $\mathbb{C}_i^{-1}$ $(i = 0,\ldots,N)$ are large enough, the second bracketed term above can be at most equal to a fraction of the first bracketed term, thus for some λ: $0 < \lambda < 1$,

$$\sup_{\left\| \begin{bmatrix} r^1 \\ s^1 \end{bmatrix} \right\| = 1} \left| a\left(\begin{bmatrix} r^1 \\ s^1 \end{bmatrix}, \begin{bmatrix} r^2 \\ s^2 \end{bmatrix} \right) \right| \geq \lambda \cdot \text{the first bracketed term in (6.11)}$$

$$\geq \gamma > 0, \quad \text{for some } \gamma.$$

Therefore

$$\inf_{\left\| \begin{bmatrix} r^2 \\ s^2 \end{bmatrix} \right\| = 1} \sup_{\left\| \begin{bmatrix} r^1 \\ s^1 \end{bmatrix} \right\| = 1} \left| a\left(\begin{bmatrix} r^1 \\ s^1 \end{bmatrix}, \begin{bmatrix} r^2 \\ s^2 \end{bmatrix} \right) \right| \geq \gamma > 0.$$

Hence (A3) and (A4) are justifiable under the assumption. In fact, the above argument shows that assumptions (A3) and (A4) are related to the earlier assumption (A2). □

Theorem 6.2 Let $(\hat{p}_{0h}, \hat{p}_h)$ be the solution of (6.9) and let S_h be a $(\tau,1)$-system. Assume that $C_i(t), z_i(t)$, $i = 1,\ldots,N$ are sufficiently smooth. Under (A1)-(A4), we have

$$(6.12)\qquad \|\hat{p}_0 - \hat{p}_{0h}\|_{H^1_{0n}} + \|\hat{p} - \hat{p}_h\|_{[H^1_{0n}]^N} \le Kh^{\mu}[\|\hat{p}_0\|_{H^\ell_n} + \|\hat{p}\|_{[H^\ell_n]^N}]$$

$$(6.13)\qquad \|\hat{p}_0 - \hat{p}_{0h}\|_{L^2_n} + \|\hat{p} - \hat{p}_h\|_{[L^2_n]^N} \le Kh^{\mu+1}[\|\hat{p}_0\|_{H^\ell_n} + \|\hat{p}\|_{[H^\ell_n]^N}]$$

provided $(\hat{p}_0,\hat{p}) \in [H^1_{0n} \cap H^\ell_n] \times [H^1_{0n} \cap H^\ell_n]^N$, where $\mu = \min(\tau-1,\ell-1)$ and $K_1 > 0$ is a constant independent of $(\hat{p}_0,\hat{p})$. Consequently,

$$(6.14)\qquad |L(\hat{p}_0,\hat{p}) - L(\hat{p}_{0h},\hat{p}_h)| \le K_2h^{2\mu}[\|\hat{p}_0\|^2_{H^\ell_n} + \|\hat{p}\|^2_{[H^\ell_n]^N}]$$

holds for some $K_2 > 0$ independent of $(\hat{p}_0,\hat{p})$.

Proof: Because $(\hat{p}_{0h},\hat{p}_h)$ satisfies (6.10) and $(\hat{p}_0,\hat{p})$ satisfies (6.7), we get

$$a\left(\begin{bmatrix}\hat{p}_0 - \hat{p}_{0h}\\ \hat{p} - \hat{p}_h\end{bmatrix}, \begin{bmatrix}r_h\\ s_h\end{bmatrix}\right) = 0, \qquad \forall (r_h,s_h) \in S_h \times [S_h]^N.$$

Therefore ([1,p. 186]) by (A3) and (A4), one gets

$$\|(\hat{p}_0 - \hat{p}_{0h}, \hat{p} - \hat{p}_h)\|_{H^1_{0n}\times[H^1_{0n}]^N} \le \left(1 + \frac{C}{\gamma}\right) \inf_{(r_h,s_h)\in S_h\times[S_h]^N} [\|\hat{p}_0 - r_h\|_{H^1_{0n}} + \|\hat{p} - s_h\|_{[H^1_{0n}]^N}].$$

for some $C > 0$ independent of h.

Using (6.8), we get (6.12).

To prove (6.13), we use Nitsche's trick ([4], [10]). By (A3) and [1], for any $g \in L^2_n \times [L^2_n]^N$, we have a unique $w(g) \in H^1_{0n} \times [H^1_{0n}]^N$ such that

$$a(w(g),y) = \langle g,y\rangle_{L_n^2 \times [L_n^2]^N}, \quad \forall y \in H_{0n}^1 \times [H_{0n}^1]^N.$$

Furthermore, we have $w(g) \in [H_{0n}^1 \cap H_n^2] \times [H_{0n}^1 \cap H_n^2]^N$, provided that $C_i(t)$ and $z_i(t)$, $i = 1,2,\ldots,N$, are sufficiently smooth. (This $w(g)$ can be obtained explicitly from integration by parts and it satisfies an equation similar to (4.7)). It is not difficult to verify that

$$\|w(g)\|_{H_n^2 \times [H_n^2]^N} \le K'\|g\|_{L_n^2 \times [L_n^2]^N},$$

where K' is independent of g. By the very same proof of the Aubin-Nitsche lemma [4, p. 137], which remains valid under (A3) and (A4), we get

$$(6.15) \qquad \|\hat{p}_0 - \hat{p}_{0h}\|_{L_n^2} + \|\hat{p} - \hat{p}_h\|_{[L_n^2]^N} \le Ch^{\mu}[\|\hat{p}_0\|_{H_n^{\ell}} + \|\hat{p}\|_{[H_n^{\ell}]^N}].$$

$$\sup_{g \in L_n^2 \times [L_n^2]^N} [\frac{1}{\|g\|} \inf_{\zeta_h \in S_h \times [S_h]^N} \|w(g) - \zeta_h\|].$$

But, by (6.8),

$$\frac{1}{\|g\|_{L_n^2 \times [L_n^2]^N}} \inf_{\zeta_h \in S_h \times [S_h]^N} \|w(g) - \zeta_h\| \le \frac{1}{\|g\|} \cdot K''h\|w(g)\|_{H_n^2}$$

$\le \frac{1}{\|g\|} \cdot K'' \cdot h \cdot K'\|g\| = K'K''h$, for some $K'' > 0$ independent of g and $w(g)$.

Using the above in (6.15), we get (6.13).

To show (6.14), we note that

$$L(\hat{p}_{0h},\hat{p}_h) - L(\hat{p}_0,\hat{p}) = 2[a(\begin{bmatrix}\hat{p}_0\\ \hat{p}\end{bmatrix},\begin{bmatrix}\hat{p}_{0h}-\hat{p}_0\\ \hat{p}_h-\hat{p}\end{bmatrix}) - \theta(\begin{bmatrix}\hat{p}_{0h}-\hat{p}_0\\ \hat{p}_h-\hat{p}\end{bmatrix})]$$

$$+ a(\begin{bmatrix}\hat{p}_{0h}-\hat{p}_0\\ \hat{p}_h-\hat{p}\end{bmatrix},\begin{bmatrix}\hat{p}_{0h}-\hat{p}_0\\ \hat{p}_h-\hat{p}\end{bmatrix}).$$

The first term on the right above is zero because of (6.7). The second term on the right can be estimated by using (6.12). Hence we get (6.14). □

<u>Corollary 6.3</u> Let

(6.16) $$\hat{x}_h \equiv E_0^{-1}(\dot{\hat{p}}_{0h} + A^*\hat{p}_{0h} + \sum_{i=1}^{N} C_i^* z_i)$$

(6.17) $$\hat{u}_{h,i} \equiv M_i^{-1}B_i^*(\hat{p}_{0h} + \sum_{j=1}^{N} \hat{p}_{h,j} - \hat{p}_{h,i}); \qquad i = 1,2,\ldots,N,$$

(6.18) $$\hat{x}_h^i \equiv -E_i^{-1}(\dot{\hat{p}}_{h,i} + A^*\hat{p}_{h,i} - C_i^* z_i) \quad ; \qquad i = 1,2,\ldots,N,$$

(6.19) $$\hat{v}_{h,i} \equiv -M_i^{-1}B_i^*\hat{p}_{h,i} \quad ; \qquad i = 1,2,\ldots,N,$$

and

$$\hat{X}_h \equiv (\hat{x}_h^1,\ldots,\hat{x}_h^N),\ \hat{v}_h \equiv (\hat{v}_{h,1},\ldots,\hat{v}_{h,N}),\ \hat{u}_h \equiv (\hat{u}_{h,1},\ldots,\hat{u}_{h,N}).$$

Then

(6.20) $$\|\hat{u} - \hat{u}_h\|_{L_n^2} + \|\hat{v} - \hat{v}_h\|_{[L_n^2]^N} \le K_3 h^{\mu+1}[\|\hat{p}_0\|_{H_n^\ell} + \|\hat{p}\|_{[H_n^\ell]^N}] \quad ,$$

$$(6.21) \qquad \|\hat{x} - \hat{x}_h\|_{L_n^2} + \|\hat{X} - \hat{X}_h\|_{[L_n^2]^N} \le K_3 h^{\mu}[\|\hat{p}_0\|_{H_n^{\ell}} + \|\hat{p}\|_{[H_n^{\ell}]^N}].$$

for some $K_3 > 0$ independent of $\hat{x}$, $\hat{u}$, $\hat{X}$, $\hat{v}$, $\hat{p}_0$ and $\hat{p}$. □

The convergence rate (6.20) is the sharpest possible [10]. The rate (6.21) is not optimal. To obtain a faster rate of convergence for x and X, one can use $\hat{u}_h$ and $\hat{v}_h$ in $(DE) = 0$ and $(DE)_i = 0$ $(1 \le i \le N)$ to solve for more accurate x and X.

§7. Examples and Computational Results

In this section, we apply the finite element method to some examples and present our numerical results.

Example 1 We consider the following two person non zero-sum game

$$\begin{cases} \begin{cases} \dot{x}(t) = x(t) + u_1(t) + 2u_2(t) + 1 & , t \in [0,T], \quad T = \pi/4, \\ x(0) = 0 \end{cases} \\ J_1(x,u) = \int_0^T \left[|x(t) + (\cos t + \tfrac{1}{2})|^2 + \tfrac{1}{2}\, |u_1(t)|^2 \right] dt \\ J_2(x,u) = \int_0^T \left[|x(t) - \sin t|^2 + 2\, |u_2(t)|^2 \right] dt. \end{cases}$$

The Lagrangian L in (4.3) corresponding to this problem is

$$\begin{aligned} (7.1)\quad L(p_0,p_1,p_2) = & -\frac{1}{2} \langle \dot{p}_0 + p_0, \frac{1}{2}(\dot{p}_0 + p_0)\rangle + \frac{1}{2}[\langle \dot{p}_1 + p_1, \dot{p}_1 + p_1\rangle + \langle \dot{p}_2 + p_2, \dot{p}_2 + p_2\rangle] \\ & - \frac{1}{2}\langle p_0 + p_1 + p_2, 4(p_0 + p_1 + p_2)\rangle + \langle p_0 + p_1 + p_2, 2\cdot p_1 + 2\cdot p_2\rangle \\ & - \langle \dot{p}_0 + p_0, \frac{1}{2}[(\cos t + \frac{1}{2}) + \sin t]\rangle - [\langle \dot{p}_1 + p_1, \cos t + \frac{1}{2}\rangle + \langle \dot{p}_2 + p_2, \sin t\rangle] \\ & - \langle p_0 + p_1 + p_2, 1\rangle - \frac{1}{2}\langle \frac{1}{2}[(\cos t + \frac{1}{2}) + \sin t], (\cos t + \frac{1}{2}) + \sin t\rangle \\ & + \frac{1}{2}[\langle \cos t + \frac{1}{2}, \cos t + \frac{1}{2}\rangle + \langle \sin t, \sin t\rangle]. \end{aligned}$$

Using $\mathbb{C}_0^{-1} = \frac{1}{2}$, $\mathbb{C}_1^{-1} = 1$, $\mathbb{C}_2^{-1} = 1$, $B_1M_1^{-1}B_1^* = 2$, $B_2M_2^{-1}B_2^* = 2$, $S = 4$, we easily verify that (A1) - (A4) are all satisfied for all $T > 0$.

We choose a (4,1)-system of Hermite cubic splines as in [13,p.56]. The interval $[0,T]$ is divided into N equal subintervals, each with mesh length $h = \frac{T}{N}$. The matrix

M_h is a $(6N+3) \times (6N+3)$ matrix. We use the IMSL high accuracy subroutine LEQ2S to solve the matrix equation $\bar{M}_h \bar{\gamma}_h = \bar{\theta}_h$ on an IBM370/Model 3033 at the Pennsylvania State University.

Numerical results are plotted in Figures 1 - 4:

(i) Figure 1: Strategy u_1 is plotted, using $h = \frac{\pi}{4}/16, \frac{\pi}{4}/32, \frac{\pi}{4}/64$, respectively. Numerical results for v_1 are found to be identical with u_1, as indicated in Theorem 4.7.

(ii) Figure 2: Strategy u_2 is plotted, using $h = \frac{\pi}{4}/16, \frac{\pi}{4}/32, \frac{\pi}{4}/64$, respectively. Numerical results for v_2 are identical with u_2.

(iii) Figure 3: State x is plotted, using $h = \frac{\pi}{4}/16, \frac{\pi}{4}/32, \frac{\pi}{4}/64$.

(iv) Figure 4: x, x^1 and x^2 are plotted, with $h = \frac{\pi}{4}/16$. Except near $t = 0$ and $t = T$ (where all three trajectories exhibit a great deal of roughness), the numerical data of x, x^1 and x^2 differ very little.

The values of $L(p_0,p_1,p_2)$ and $J(x,u;X,v)$ are found to be

$$L = J = 0.02394619\,, \qquad h = \frac{\pi}{4}/16$$

(7.2) $$L = J = 0.01211985\,, \qquad h = \frac{\pi}{4}/32$$

$$L = J = 0.00609733\,, \qquad h = \frac{\pi}{4}/64.$$

A quick observation points out that L converges to 0 with rate $\mathcal{O}(h^1)$. This seems to contradict (6.14), which predicts that the rate should be $\mathcal{O}(h^6)$. Nevertheless, we believe that this is not really paradoxical because, first of all, $\mathcal{O}(h^6)$ is a quite high rate of convergence, which is hard to verify and, secondly, we believe that the values of L and J in (7.2) are probably composed of quadrature and round off errors, since our h is very small and the matrix solver has high accuracy. All of our calculations were carried out with double precision.

In Table 1, we list some values of u_1, u_2, x, x^1, x^2, p_0, p_1 and p_2 at certain selected nodal points.

Example 2 We compute Example 1 again, but with $T = 2\pi$ and $h = 2\pi/16$. The graphs for u_1 and u_2 are plotted in Figure 5. Here again we have $v_1 = u_1$, $v_2 = u_2$ in numerical values. The graphs for x, x^1 and x^2 are plotted in Figure 6. The reader may compare them with the pictures of Example 1.

Example 3 We consider the following 2-person non-zero sum game:

$$\begin{cases} \begin{cases} \dot{x}(t) = x(t) + \cos t \cdot u_1(t) + \sin t \cdot u_2(t) + 1, & 0 \le t \le T, \\ x(0) = 0, \end{cases} \\ J_1(x,u) \equiv \int_0^T [|x(t) - d_1(\cos t + \frac{1}{2})|^2 + \frac{1}{3} u_1^2(t)]dt, \\ \\ J_2(x,u) \equiv \int_0^T [|x(t) - d_2 \sin t|^2 + \frac{1}{2} u_2^2(t)]dt, \end{cases}$$

It is not clear to us whether conditions (A2) - (A4) are satisfied when T is large.

For $(d_1,d_2) = (-1,1)$ and $T = \frac{\pi}{4}$, we find that

$L = 0.02394619$, $h = \frac{\pi}{4}/16$

$L = 0.01211985$, $h = \frac{\pi}{4}/32$

$L = 0.00609733$, $h = \frac{\pi}{4}/64$.

Surprisingly, they agree identically with the values in (7.2) (except the last few digits which have been rounded off by us).

For $T = 2\pi$, $(d_1,d_2) = (-1, 0.9)$, we find that

$L = -0.02630621$, $h = 2\pi/4$

$L = -0.03772221$, $h = 2\pi/8$

$L = -0.04121127$, $h = 2\pi/16$

$L = -0.04456356$, $h = 2\pi/32$

$L = -0.05005449$, $h = 2\pi/64$

These values of L are all negative and seem to be divergent. See [3, §4, Example 3] for further discussions. □

Due to the lack of any known closed form solutions to make comparisons, error estimates (6.20) and (6.21) can not be verified at this stage. However, in Part II [3] of our papers, numerical results for Example 1 will be compared with those obtained from another very different approach - the penalty method. They manifest remarkable agreement. This gives a good indication that our treatment and calculations are sound.

Note added in proof: We have recently improved the order of convergence of L to $\mathcal{O}(h^5)$, which is close to the predicted rate $\mathcal{O}(h^6)$ mentioned at the last paragraph of page 59. In addition, the roughness of the state x as well as x^1 and x^2 as shown in Figures 3, 4, and 6, and those in certain figures in Part II of our papers, have all been eliminated. The improved numerical results will be published later on in a technical journal.

	$t = \frac{1}{4} \cdot \frac{\pi}{4}$			$t = \frac{1}{2} \cdot \frac{\pi}{4}$			$t = \frac{3}{4} \cdot \frac{\pi}{4}$			$t = \frac{\pi}{4} = T$		
	$h = \frac{\pi}{4}/16$	$h = \frac{\pi}{4}/32$	$h = \frac{\pi}{4}/64$	$h = \frac{\pi}{4}/16$	$h = \frac{\pi}{4}/32$	$h = \frac{\pi}{4}/64$	$h = \frac{\pi}{4}/16$	$h = \frac{\pi}{4}/32$	$h = \frac{\pi}{4}/64$	$h = \frac{\pi}{4}/16$	$h = \frac{\pi}{4}/32$	$h = \frac{\pi}{4}/64$
u_1	-2.022507	-2.050318	-2.064450	-1.199239	-1.219113	-1.229223	-0.536199	-0.549396	-0.556116	0.0	0.0	0.0
u_2	0.421594	0.431190	0.436094	0.271250	0.278159	0.281693	0.123312	0.127565	0.129746	0.0	0.0	0.0
x	-0.131033	-0.127947	-0.126924	-0.138533	-0.137644	-0.137191	-0.053930	-0.053681	-0.053709	-0.25000	-0.250000	-0.250000
x^1	-0.145114	-0.134294	-0.130115	-0.153802	-0.145366	-0.141075	-0.073320	-0.063079	-0.058435	-1.207107	-1.207107	-1.207107
x^2	-0.116951	-0.121600	-0.123733	-0.123263	-0.129922	-0.133308	-0.034540	-0.044283	-0.048983	0.707107	0.707107	0.707107
p_0	0.589659	-0.593969	-0.596131	-0.328370	-0.331397	-0.332918	-0.144788	-0.147134	-0.148312	0.0	0.0	0.0
p_1	1.011253	1.025159	1.032225	0.599620	0.609556	0.614612	0.268099	0.274698	0.278058	0.0	0.0	0.0
p_2	-0.421594	-0.431190	-0.436094	0.271250	-0.278159	-0.281694	-0.123312	-0.127565	-0.129746	0.0	0.0	0.0

Remark: The numerical values of v_1, v_2 are identical, respectively, with u_1, u_2. All entries above are rounded off figures with six decimal place accuracy.

Table 1: Numerical Values of u_1, u_2, x, x^1, x^2, p_0, p_1 and p_2 at $t = \frac{1}{4} \cdot \frac{\pi}{4}$, $\frac{1}{2} \cdot \frac{\pi}{4}$, $\frac{3}{4} \cdot \frac{\pi}{4}$, and $\frac{\pi}{4}$.

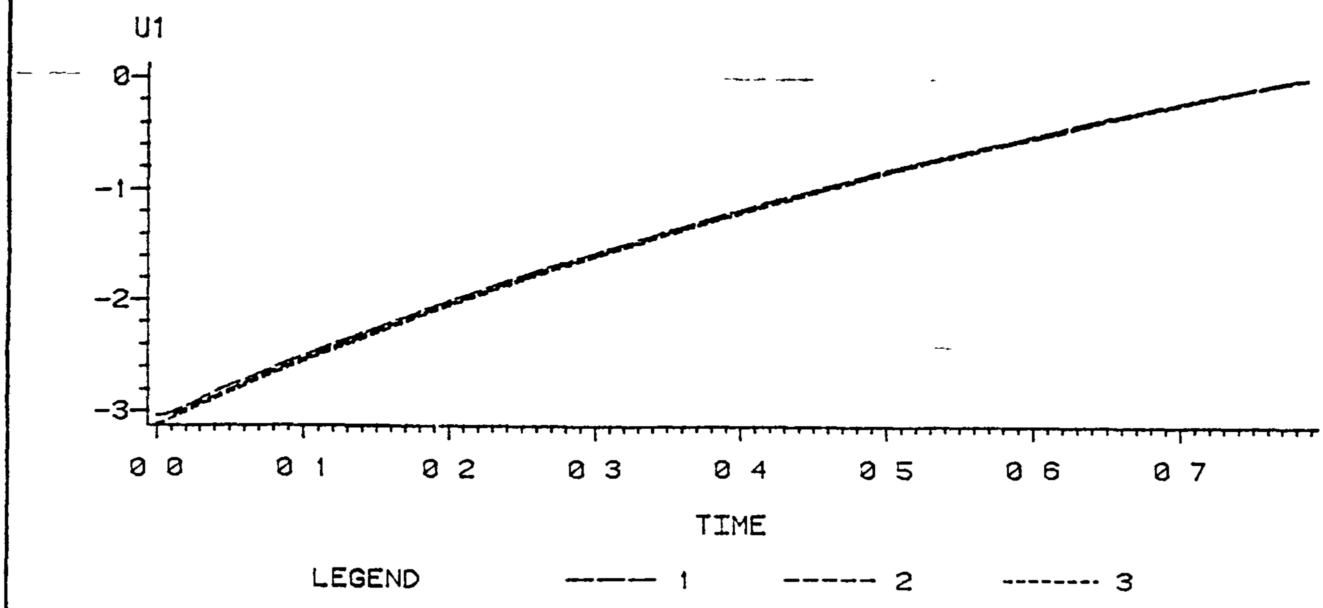


Figure 1: Duality Solution of u_1, Example 1

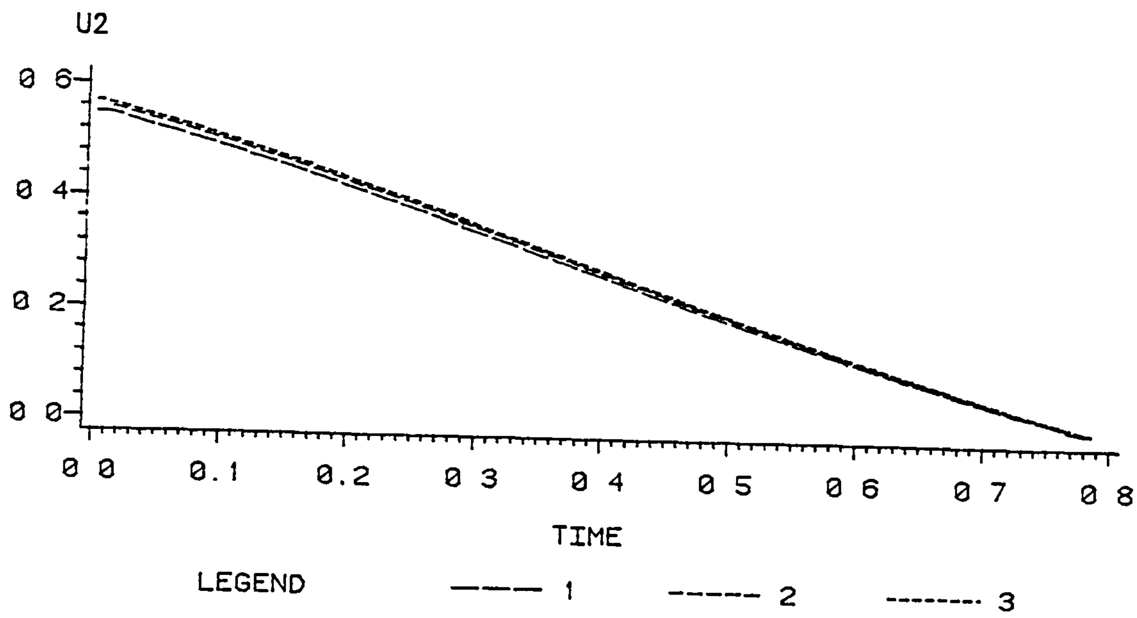


Figure 2: Duality Solution of u_2, Example 1

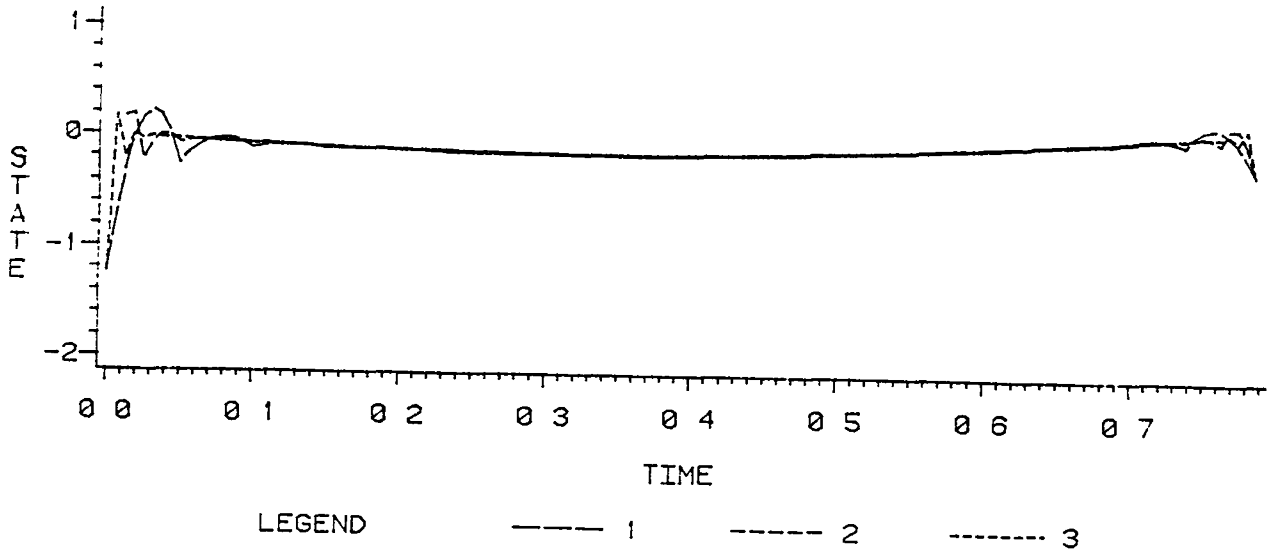
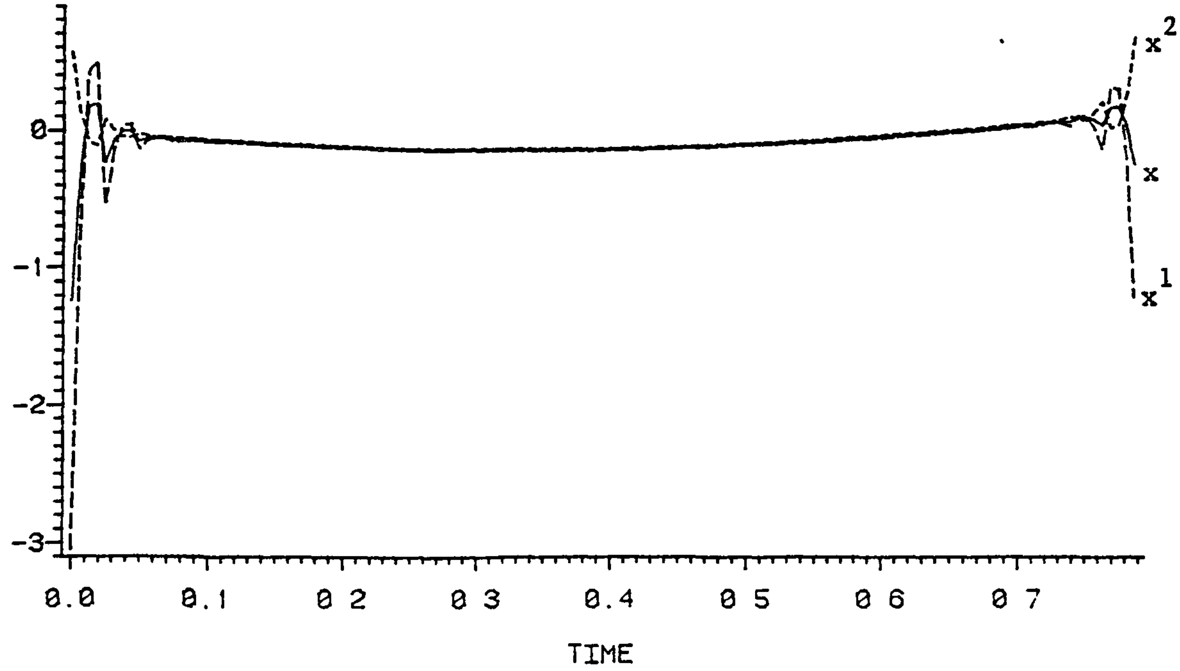


Figure 3: Duality Solution of x, Example 1

Throughout Figures 1, 2 and 3, curves 1, 2, and 3 represent the numerical solutions with $h = \frac{\pi}{4}/16$, $\frac{\pi}{4}/32$ and $\frac{\pi}{4}/64$, respectively, for Example 1.



<u>Figure 4</u>: Duality Solutions of x, x^1 and x^2, Example 1, with $h = \frac{\pi}{4}/16$.

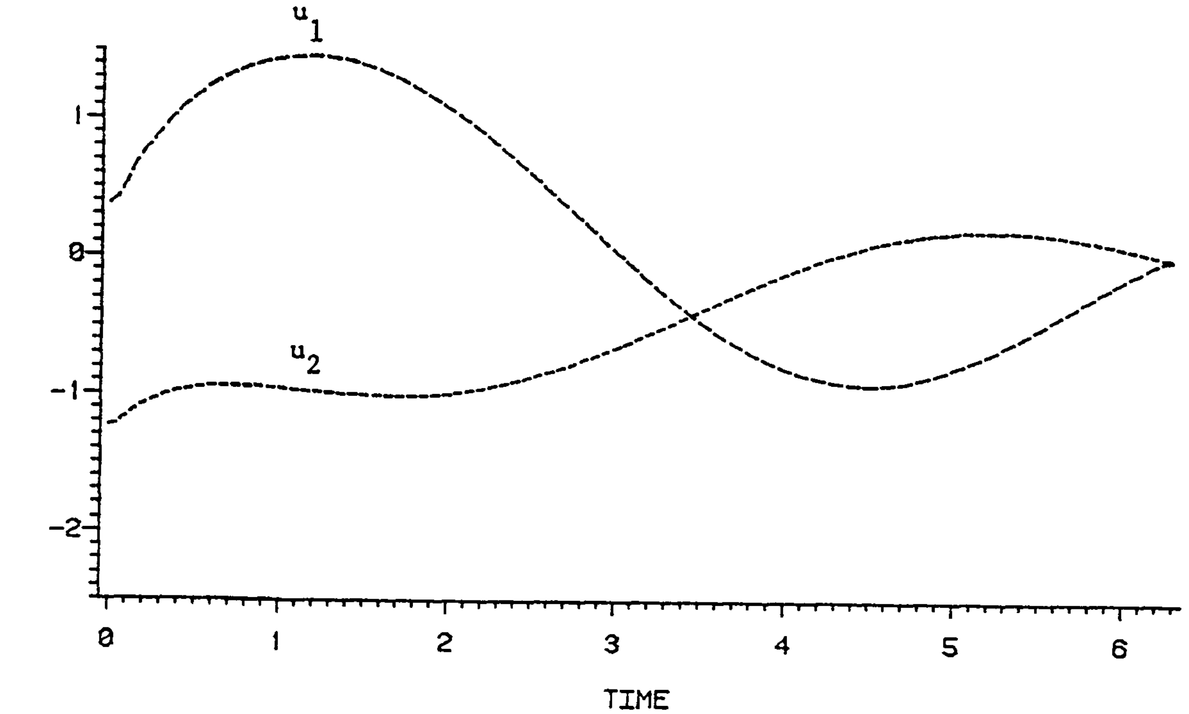


Figure 5: The Case $T = 2\pi$, Strategies u_1 and u_2, Example 2, with $h = 2\pi/16$.

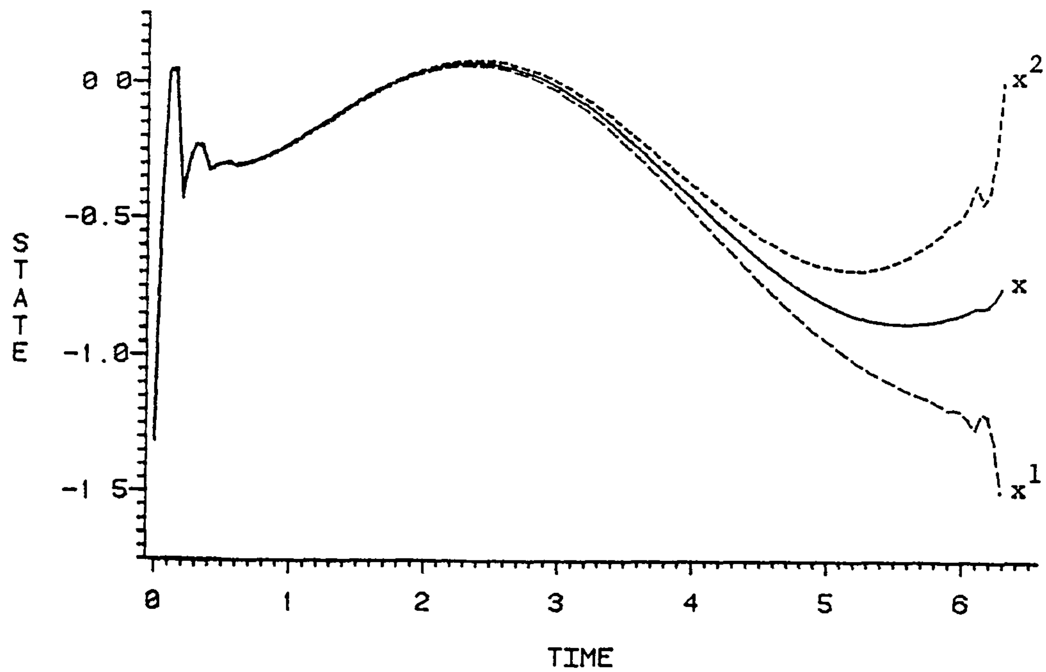


Figure 6: The Case $T = 2\pi$, State x, x^1 and x^2, Example 2, with $h = 2\pi/16$.

REFERENCES

[1] I. Babuska and A.K. Aziz, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A.K. Aziz, ed., Academic Press, New York, 1972.

[2] W.E. Bosarge and O.G. Johnson, Error bounds of high order accuracy for the state regulator problem via piecewise polynomial approximation, SIAM J. Control, 9, 15-28, 1971.

[3] G. Chen, W. H. Mills, Q. Zheng, W. Shaw, N-person differential games, Part II, the penalty method, NASA Contractor Report No. 166111, NASA Langley Research Center, Hampton, Va 23665, April 1983.

[4] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North Holland, Amsterdam, 1978.

[5] K. Fan, Sur un théorème minimax, C.R. Acad. Sci., Paris, 259, 3925-3928, 1964.

[6] A. Friedman, Differential Games, Wiley-Interscience, New York, 1971.

[7] W.W. Hager and S.K. Mitter, Lagrange duality theory for convex control problems, SIAM J. Control Opt., 14, 843-856, 1976.

[8] R. Issacs, Differential Games, Wiley, New York, 1965.

[9] D.L. Lukes and D.L. Russell, A global theory for linear quadratic differential games, J. Math. Anal. Appl., 33, 96-123, 1971.

[10] F.H. Mathis and G.W. Reddien, Ritz-Trefftz approximation in optimal control, SIAM J. Control Opt., 17, 307-310, 1979.

[11] J. Ponstein, Approaches to the theory of optimization, Cambridge Univ. Press, London, 1980.

[12] D. L. Russell, Mathematics of Finite Dimensional Control Systems, Theory and Design, Marcel Dekker, New York, 1979.

[13] G. Strang and G. Fix, An Analysis of the Finite Element Method, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

1 Report No	2 Government Accession No	3 Recipient's Catalog No
NASA CR-166110		

4 Title and Subtitle	5 Report Date
N-PERSON DIFFERENTIAL GAMES PART I: DUALITY-FINITE ELEMENT METHODS	April 1983
	6 Performing Organization Code
7 Author(s) Goong Chen and Quan Zheng	8 Performing Organization Report No 83-7
	10 Work Unit No
9 Performing Organization Name and Address INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING MAIL STOP 132C, NASA LANGLEY RESEARCH CENTER HAMPTON, VA 23665	11 Contract or Grant No NAS1-15810
	13 Type of Report and Period Covered contractor report
12 Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546	14 Sponsoring Agency Code

15 Supplementary Notes Additional support: NSF Grant MCS 81-01892
Technical monitor: Robert H. Tolson
Final Report

16 Abstract Standard theory of differential games focuses the study on two-person zero-sum games, and treat N-person games separately and differently. In this paper we present a new equivalent formulation of the Nash equilibrium strategy for N-person differential games. Our contributions are the following:

1) Our min-max formulation unifies the study of two-person zero-sum with that of the general N-person non zero-sum games. Indeed, it opens a new avenue of systematic research for differential games.
2) We are successful in applying the finite element method to compute solutions of linear-quadratic N-person games. We have also established numerical error estimates. Our calculations, which are based upon the dual formulation, are very efficient.
3) We are able to establish global existence and uniqueness of solutions of the Riccati equation in our form, which is important in synthesis. This, to our knowledge, has not been done elsewhere by any other researchers.

This paper's particular emphasis is on the duality approach, which is motivated by computational needs and is done by introducing N + 1 Language multipliers: one for each player and one "joint multiplier" for all players. For N-person linear quadratic games, we show that under suitable conditions the primal min-max problem is equivalent to its dual min-max problem, which is acutally a saddle point and is computed by finite elements. Numerical examples are presented in the last section.

17 Key Words (Suggested by Author(s))	18 Distribution Statement
differential games, Riccati equation, finite elements	Unclassified-Unlimited Subject Category 64

19 Security Classif (of this report)	20 Security Classif (of this page)	21 No of Pages	22 Price
Unclassified	Unclassified	69	A04

End of Document